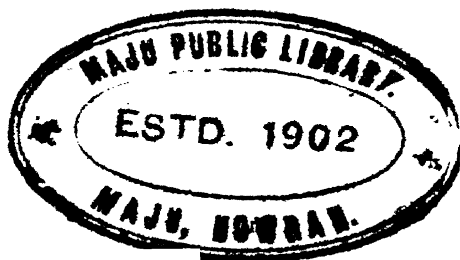
ALGEBRA FOR BEGINNERS

WITH NUMEROUS EXAMPLES



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PREFACE.

The present work has been undertaken at the request of many teachers, in order to be placed in the hands of beginners, and to serve as an introduction to the larger treatise published by the author; it is accordingly based on the earlier chapters of that treatise, but is of a more elementary character. Great pains have been taken to render the work intelligible to young students, by the use of simple language and by copious explanations.

In determining the subjects to be included and the space to be assigned to each, the author has been guided by the papers given at the various examinations in elementary Algebra which are now carried on in this country. The book may be said to consist of three parts. The first part contains the elementary operations in integral and fractional expressions; it occupies eighteen chapters. The second part contains the solution of equations and problems; it occupies twelve chapters. The subjects contained in these two parts constitute nearly the whole of every examination paper which was consulted, and accordingly they are treated with ample detail of illustration and exercise. The third part forms the remainder of the book; it consists of various subjects which are introduced but rarely into the examination papers, and which are therefore more briefly discussed.

The subjects are arranged in what appears to be the most natural order. But many teachers find it advantageous to introduce easy equations and problems at a very early stage, and accordingly provision has been made for

such a course. It will be found that Chapters XIX. and XXI. may be taken as soon as a student has proceeded as far as algebraical multiplication.

In accordance with the recommendation of teachers, the examples for exercise are very numerous. Some of these have been selected from the College and University examination papers, and some from the works of Saunderson and Simpson; many however are original, and are constructed with reference to points which have been shewn to be important by the author's experience as a teacher and an examiner.

The author has to acknowledge the kindness of many distinguished teachers who have examined the sheets of his work and have given him valuable suggestions. Any remarks on the work, and especially the indication of difficulties either in the text or the examples, will be most thankfully received.

I. TODHUNTER.

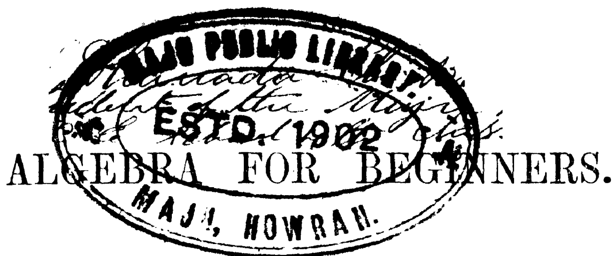
ST JOHN'S COLLEGE,
July 1863.

Four new Chapters have been added to the present edition, and also a collection of Miscellaneous Examples which are arranged in sets, each set containing ten examples. These additions have been made at the request of some eminent teachers, in order to increase the utility of the work.

July 1867.



CONTENTS



ALGEBRA FOR BEGINNERS.

I. *The Principal Signs.*

1. ALGEBRA is the science in which we reason about numbers, with the aid of letters to denote the numbers, and of certain signs to denote the operations performed on the numbers, and the relations of the numbers to each other.

2. Numbers may be either *known* numbers, or numbers which have to be found, and which are therefore called *unknown* numbers. It is usual to represent *known* numbers by the first letters of the alphabet, a, b, c, &c., and *unknown* numbers by the last letters x, y, z; this is however not a necessary rule, and so need not be strictly obeyed. Numbers may be either whole or fractional. The word *quantity* is often used with the same meaning as *number*. The word *integer* is often used instead of *whole number*.

3. The beginner has to accustom himself to the use of letters for representing numbers, and to learn the meaning of the signs; we shall begin by explaining the most important signs and illustrating their use. We shall assume that the student has a knowledge of the elements of Arithmetic, and that he admits the truth of the common notions required in all parts of mathematics, such as, *if equals be added to equals the wholes are equal*, and the like.

4. The sign + placed before a number denotes that the number is to be *added*. Thus $a + b$ denotes that the number represented by b is to be added to the number repre-

sented by a. If a represent 9 and b represent 3, then $a+b$ represents 12. The sign + is called the *plus sign*, and $a+b$ is read thus "a *plus* b."

5. The sign − placed before a number denotes that the number is to be *subtracted*. Thus $a-b$ denotes that the number represented by b is to be subtracted from the number represented by a. If a represent 9 and b represent 3, then $a-b$ represents 6. The sign − is called the *minus sign*, and $a-b$ is read thus "a *minus* b."

6. Similarly $a+b+c$ denotes that we are to add b to a, and then add c to the result; $a+b-c$ denotes that we are to add b to a, and then subtract c from the result; $a-b+c$ denotes that we are to subtract b from a, and then add c to the result; $a-b-c$ denotes that we are to subtract b from a, and then subtract c from the result.

7. The sign = denotes that the numbers between which it is placed are *equal*. Thus $a=b$ denotes that the number represented by a is equal to the number represented by b. And $a+b=c$ denotes that the sum of the numbers represented by a and b is equal to the number represented by c; so that if a represent 9, and b represent 3, then c must represent 12. The sign = is called the *sign of equality*, and $a=b$ is read thus "a *equals* b" or "a *is equal to* b."

8. The sign × denotes that the numbers between which it stands are to be *multiplied* together. Thus $a \times b$ denotes that the number represented by a is to be multiplied by the number represented by b. If a represent 9, and b represent 3, then $a \times b$ represents 27. The sign × is called the *sign of multiplication*, and $a \times b$ is read thus "a *into* b." Similarly $a \times b \times c$ denotes the product of the numbers represented by a, b, and c.

9. The sign of multiplication is however often omitted for the sake of brevity; thus ab is used instead of $a \times b$, and has the same meaning; so also abc is used instead of $a \times b \times c$, and has the same meaning.

The sign of multiplication must not be omitted when numbers are expressed in the ordinary way by figures. Thus 45 cannot be used to represent the product of 4 and

5, because a different meaning has already been appropriated to 45, namely, *forty-five*. We must therefore represent the product of 4 and 5 in another way, and 4×5 is the way which is adopted. Sometimes, however, a point is used instead of the sign $\times$; thus 4.5 is used instead of 4×5. To prevent any confusion between the point thus used as a sign of multiplication, and the point used in the notation for decimal fractions, it is advisable to place the point in the latter case higher up; thus 4·5 may be kept to denote $4 + \frac{5}{10}$. But in fact the point is not used instead of the sign $\times$ except in cases where there can be no ambiguity. For example, 1.2.3.4 may be put for $1 \times 2 \times 3 \times 4$ because the points here will not be taken for decimal points.

The point is sometimes placed instead of the sign $\times$ between two letters; so that $a . b$ is used instead of $a \times b$. But the point is here superfluous, because, as we have said, ab is used instead of $a \times b$. Nor is the point, nor the sign $\times$, necessary between a number expressed in the ordinary way by a figure and a number represented by a letter; so that, for example, $3a$ is used instead of $3 \times a$, and has the same meaning.

10. The sign $\div$ denotes that the number which precedes it is to be *divided* by the number which follows it. Thus $a \div b$ denotes that the number represented by a is to be divided by the number represented by b. If a represent 8, and b represent 4, then $a \div b$ represents 2. The sign $\div$ is called the *sign of division*, and $a \div b$ is read thus "a *by* b."

There is also another way of denoting that one number is to be divided by another; the dividend is placed over the divisor with a line between them. Thus $\frac{a}{b}$ is used instead of $a \div b$, and has the same meaning.

11. The letters of the alphabet, and the signs which we have already explained, together with those which may occur hereafter, are called *algebraical symbols*, because they are used to represent the numbers about which we may be reasoning, the operations performed on them, and

their relations to each other. Any collection of Algebraical symbols is called an *algebraical expression*, or briefly an *expression*.

12. We shall now give some examples as an exercise in the use of the symbols which have been explained; these examples consist in finding the numerical values of certain algebraical expressions.

Suppose $a=1$, $b=2$, $c=3$, $d=5$, $e=6$, $f=0$. Then

$$7a+3b-2d+f=7+6-10+0=13-10=3.$$

$$2ab+8bc-ae+df=4+48-6+0=52-6=46.$$

$$\frac{4ac}{b}+\frac{10be}{cd}-\frac{de}{ac}=\frac{12}{2}+\frac{120}{15}-\frac{30}{3}=6+8-10=14-10=4.$$

$$\frac{4c+5e}{d-b}=\frac{12+30}{5-2}=\frac{42}{3}=14.$$

Examples. I.

If $a=1$, $b=2$, $c=3$, $d=4$, $e=5$, $f=0$, find the numerical values of the following expressions:

1. $9a+2b+3c-2f$.
2. $4e-3a-3b+5c$.
3. $7ae+3bc+9d-af$.
4. $8abc-bcd+9cde-def$.
5. $abcd+abce+abde+acde+bcde$.
6. $\frac{4a}{b}+\frac{9b}{c}+\frac{8c}{d}-\frac{5d}{e}$.
7. $\frac{4ac}{b}+\frac{8bc}{d}-\frac{5cd}{e}$.
8. $\frac{12a}{bc}+\frac{6b}{cd}+\frac{20c}{de}$.
9. $\frac{cde}{ab}+\frac{5bcd}{ae}-\frac{6ade}{bc}$.
10. $7e+bcd-\frac{3bde}{2ac}$.
11. $\frac{2a+5b}{c}+\frac{3b+2c}{d}-\frac{a+b+c+d}{2e}$.
12. $\frac{b+c+3e}{e+c-d}$.
13. $\frac{a+c}{c-a}+\frac{b+d}{d-b}+\frac{c+e}{e-c}$.
14. $\frac{a+b+c+d+e}{e-d+c-b+a}$.

II. *Factor. Coefficient. Power. Terms.*

13. When one number consists of the product of two or more numbers, each of the latter is called a *factor* of the product. Thus, for example, $2 \times 3 \times 5 = 30$; and each of the numbers 2, 3, and 5 is a *factor* of the product 30. Or we may regard 30 as the product of the two factors, 2 and 15, or as the product of the two factors 6 and 5, or as the product of the two factors 3 and 10. And so, also, we may consider $4ab$ as the product of the two factors 4 and ab, or as the product of the two factors $4a$ and b, or as the product of the two factors $4b$ and a; or we may regard it as the product of the three factors 4 and a and b.

14. When a number consists of the product of two factors, each factor is called the *coefficient* of the other factor; so that *coefficient* is equivalent to *co-factor.* Thus considering $4ab$ as the product of 4 and ab, we call 4 the coefficient of ab, and ab the coefficient of 4; and considering $4ab$ as the product of $4a$ and b, we call $4a$ the coefficient of b, and b the coefficient of $4a$. There will be little occasion to use the word coefficient in practice in any of these cases except the first, that is the case in which 4 is regarded as the coefficient of ab; but for the sake of distinctness we speak of 4 as the *numerical coefficient* of ab in $4ab$, or briefly as the *numerical coefficient.* Thus when a product consists of one factor which is represented *arithmetically*, that is by a figure or figures, and of another factor which is represented *algebraically*, that is by a letter or letters, the former factor is called the *numerical coefficient.*

15. When all the factors of a product are equal, the product is called a *power* of that factor. Thus 7×7 is called the *second power* of 7; $7 \times 7 \times 7$ is called the *third power* of 7; $7 \times 7 \times 7 \times 7$ is called the *fourth power* of 7; and so on. In like manner $a \times a$ is called the *second power* of a; $a \times a \times a$ is called the *third power* of a; $a \times a \times a \times a$ is called the *fourth power* of a; and so on. And a itself is sometimes called the *first power* of a.

16. A power is more briefly denoted thus: instead of expressing all the equal factors, we express the factor once, and place over it the number which indicates how often it is to be repeated. Thus a^2 is used to denote $a \times a$; a^3 is used to denote $a \times a \times a$; a^4 is used to denote $a \times a \times a \times a$; and so on. And a^1 may be used to denote the first power of a, that is a itself; so that a^1 has *the same meaning as* a.

17. A number placed over another to indicate how many times the latter occurs as a factor in a power, is called an *index of the power*, or an *exponent of the power;* or, briefly, an *index*, or *exponent*.

Thus, for example, in a^3 the exponent is 3; in a^n the exponent is n.

18. The student must distinguish very carefully between a *coefficient* and an *exponent*. Thus $3c$ means *three times c;* here 3 is a *coefficient*. But c^3 means *c times c times c;* here 3 is an *exponent*. That is

$$3c = c + c + c,$$

$$c^3 = c \times c \times c.$$

19. The second power of a, that is a^2, is often called the *square* of a, or *a squared;* and the third power of a, that is a^3, is often called the *cube* of a, or *a cubed*. There are no such words in use for the higher powers; a^4 is read thus "a *to the fourth power*," or briefly "a *to the fourth*."

20. If an expression contain no parts connected by the signs + and −, it is called a *simple* expression. If an expression contain parts connected by the signs + and − it is called a *compound* expression, and the parts connected by the signs + and − are called *terms* of the expression.

Thus ax, $4bc$, and $5a^2c^2$ are simple expressions; $a^2 + b^3 - c^4$ is a compound expression, and a^2, b^3, and c^4 are its terms.

21. When an expression consists of two terms it is called a *binomial* expression: when it consists of three terms it is called a *trinomial* expression; any expression consisting of several terms may be called a *multinomial* expression, or a *polynomial* expression.

Thus $2a+3b$ is a binomial expression; $a-2b+5c$ is a trinomial expression; and $a-b+c-d-e$ may be called a multinomial expression or a polynomial expression.

22. Each of the letters which occur in a term is called a *dimension* of the term, and the number of the letters is called the *degree* of the term. Thus a^2b^3c or $a\times a\times b\times b\times b\times c$ is said to be of six dimensions or of the sixth degree. A numerical coefficient is not counted; thus $9a^3b^4$ and a^3b^4 are of the same dimensions, namely seven dimensions. Thus the word *dimensions* refers to the number of algebraical multiplications involved in the term; that is, the *degree* of a term, or the *number of its dimensions*, is the *sum of the exponents of its algebraical factors*, provided we remember that if no exponent be expressed the exponent 1 must be understood, as indicated in Art. 16.

23. An expression is said to be *homogeneous* when all its terms are of the same dimensions. Thus $7a^3+3a^2b+4abc$ is homogeneous, for each term is of three dimensions.

We shall now give some more examples of finding the numerical values of algebraical expressions.

Suppose $a=1$, $b=2$, $c=3$, $d=4$, $e=5$, $f=0$. Then

$$b^2=4,\quad b^3=8,\quad b^4=16,\quad b^5=32.$$

$$3b^2=3\times 4=12,\quad 5b^3=5\times 8=40,\quad 9b^5=9\times 32=288.$$

$$e^a=5^1=5,\quad e^b=5^2=25,\quad e^c=5^3=125.$$

$$a^2b^3=1\times 8=8,\quad 3b^2c^2=3\times 4\times 9=108.$$

$$d^3+c^2-7ab+f^2=64+9-14+0=59.$$

$$\frac{3c^2-4c-10}{c^3-2c^2+5c-23}=\frac{27-12-10}{27-18+15-23}=\frac{5}{1}=5.$$

$$\frac{e^3+d^3}{e+d}-\frac{c^3-a^3}{c-a}=\frac{125+64}{5+4}-\frac{27-1}{3-1}$$

$$=\frac{189}{9}-\frac{26}{2}=21-13=8.$$

Examples. II.

If $a=1$, $b=2$, $c=3$, $d=4$, $e=5$, $f=0$, find the numerical values of the following expressions:

1. $a^2+b^2+c^2+d^2+e^2+f^2$.

2. $e^3-d^3+c^3-b^3+a^3$.

3. $abc^2+bcd^2-dea^2+f^3$.

4. $c^3-2c^2+4c-13$.

5. $a^3+3a^2b+3ab^2+b^3$.

6. $e^4+6e^2b^2+b^4-4e^3b-4eb^3$.

7. $\dfrac{b^2c^2}{4a}+\dfrac{de}{b^2}-\dfrac{32}{b^4}$.

8. $\dfrac{2e+2}{e-3}+\dfrac{3e-9}{e-2}+\dfrac{e^2-1}{e+3}$.

9. $\dfrac{a^2+b^2}{e}+\dfrac{c^2+e^2}{b}+\dfrac{e^2-d^2}{c}$.

10. $\dfrac{8a^2+3b^2}{a^2+b^2}+\dfrac{4c^2+6b^2}{c^2-b^2}-\dfrac{c^2+d^2}{e^2}$.

11. $\dfrac{28}{a^2+b^2+c^2}+\dfrac{12}{d^2-c^2-b^2}+\dfrac{4}{a^2+e^2-c^2-d^2}$.

12. $\dfrac{a^4+4a^3b+6a^2b^2+4ab^3+b^4}{a^3+3a^2b+3ab^2+b^3}$.

13. $\dfrac{d^c}{b^e}$.

14. $\dfrac{e^c+b^a}{c^b-b^c}$.

15. $\dfrac{b^e+d^c}{b^2+d^2-bd}$.

16. $\dfrac{e^c-d^c}{e^2+ed+d^2}$.

III. *Remaining Signs. Brackets.*

24. The difference of two numbers is sometimes denoted by the sign $\sim$; thus $a \sim b$ denotes the difference of the numbers represented by a and b; and is equal to $a-b$, or $b-a$, according as a is greater than b, or less than b: but this symbol $\sim$ is very rarely required.

25. The sign $>$ denotes *is greater than*, and the sign $<$ denotes *is less than*; thus $a>b$ denotes that the number represented by a is greater than the number represented by b, and $b<a$ denotes that the number represented by b is less than the number represented by a. Thus in both cases the opening of the angle is turned towards the greater number.

26. The sign $\therefore$ denotes *then* or *therefore*; the sign $\because$ denotes *since* or *because*.

27. The *square root* of any assigned number is that number which has the assigned number for its *square* or *second power*. The *cube root* of any assigned number is that number which has the assigned number for its *cube* or *third power*. The *fourth root* of any assigned number is that number which has the assigned number for its fourth power. And so on.

Thus since $49=7^2$, the square root of 49 is 7; and so if $a=b^2$, the square root of a is b. In like manner, since $125=5^3$, the cube root of 125 is 5; and so if $a=c^3$, the cube root of a is c.

28. The square root of a *may* be denoted thus $\sqrt[2]{a}$; but generally it is denoted simply thus $\sqrt{a}$. The cube root of a is denoted thus $\sqrt[3]{a}$. The fourth root of a is denoted thus $\sqrt[4]{a}$. And so on.

Thus $\sqrt{9}=3$; $\sqrt[3]{8}=2$.

The sign $\sqrt{}$ is said to be a corruption of the initial letter of the word *radix*.

29. When two or more numbers are to be treated as forming one number they are enclosed within *brackets.* Thus, suppose we have to denote that the sum of a and b is to be multiplied by c; we denote it thus $(a+b)\times c$ or $\{a+b\}\times c$, or simply $(a+b)c$ or $\{a+b\}c$; here we mean that the *whole* of $a+b$ is to be multiplied by c. Now if we omit the brackets we have $a+bc$, and this denotes that b *only* is to be multiplied by c and the result added to a. Similarly, $(a+b-c)d$ denotes that the result expressed by $a+b-c$ is to be multiplied by d, or that the *whole* of $a+b-c$ is to be multiplied by d; but if we omit the brackets we have $a+b-cd$, and this denotes that c *only* is to be multiplied by d and the result subtracted from $a+b$.

So also $(a-b+c)\times(d+e)$ denotes that the result expressed by $a-b+c$ is to be multiplied by the result expressed by $d+e$. This may also be denoted simply thus $(a-b+c)(d+e)$; just as $a\times b$ is shortened into ab.

So also $\sqrt{}(a+b+c)$ denotes that we are to obtain the result expressed by $a+b+c$, and then take the square root of this result.

So also $(ab)^2$ denotes $ab\times ab$; and $(ab)^3$ denotes $ab\times ab\times ab$.

So also $(a+b-c)\div(d+e)$ denotes that the result expressed by $a+b-c$ is to be divided by the result expressed by $d+e$.

30. Sometimes instead of using brackets a line is drawn over the numbers which are to be treated as forming one number. Thus $\overline{a-b+c}\times\overline{d+e}$ is used with the same meaning as $(a-b+c)\times(d+e)$. A line used for this purpose is called a *vinculum*. So also $(a+b-c)\div(d+e)$ may be denoted thus $\frac{a+b-c}{d+e}$; and here the line between $a+b-c$ and $d+e$ is really a *vinculum* used in a particular sense.

31. We have now explained all the signs which are used in algebra. We may observe that in some cases the word *sign* is applied specially to the two signs $+$ and $-$; thus in the Rule for Subtraction we shall speak of *changing*

the signs, meaning the signs + and −; and in multiplication and division we shall speak of the *Rule of Signs*, meaning a rule relating to the signs + and −.

32. We shall now give some more examples of finding the numerical values of expressions.

Suppose $a=1$, $b=2$, $c=3$, $d=5$, $e=8$. Then

$\sqrt{}(2b+4c)=\sqrt{}(4+12)=\sqrt{}(16)=4.$

$\sqrt[3]{}(4c-2b)=\sqrt[3]{}(12-4)=\sqrt[3]{}(8)=2.$

$e\sqrt{}(2b+4c)-(2d-b)\sqrt[3]{}(4c-2b)=8\times4-8\times2=32-16=16.$

$\sqrt{}\{(e-b)(2e-5b)\}=\sqrt{}\{(8-2)(16-10)\}=\sqrt{}(6\times6)=6.$

$\{(e-d)(b+c)-(d-c)(c+a)\}(a+d)=\{3\times5-2\times4\}6=7\times6=42.$

$\sqrt[3]{}(c^3+3c^2b+3cb^2+b^3)\div\sqrt{}(a^2+b^2-2ab)$
$=\sqrt[3]{}(27+54+36+8)\div\sqrt{}(1+4-4)=\sqrt[3]{}(125)\div1=5.$

EXAMPLES. III.

If $a=1$, $b=2$, $c=3$, $d=5$, $e=8$, find the numerical values of the following expressions:

1. $a(b+c)$.
2. $b(c+d)$.
3. $c(e-d)$.
4. $b^2(a^2+e^2-c^2)$.
5. $c^2(e^2-b^2-c^2)$.
6. $\dfrac{a^2+c^2+d^2}{a^2+b^2}$.
7. $\dfrac{9a+3d^2+e^2}{2c^2-4b^2}$.
8. $\sqrt{}(3bce)$.
9. $\sqrt{}(2b+4d+5e)$.
10. $(a+2b+3c+5e-4d)(6e-5d-4c-3b+2a)$.
11. $(a^2+b^2+c^2)(e^2-d^2-c^2)$.
12. $(3d^2-7c^2)^2$.
13. $e\sqrt{}(d^2-3e)+d\sqrt{}(d^2+3e)$.
14. $e-\{\sqrt{}(e+1)+2\}+(e-\sqrt[3]{}e)\sqrt{}(e-4)$.
15. $\sqrt{}(a^2+2ab+b^2)\times\sqrt[3]{}(a^3+3a^2b+3ab^2+b^3)$.
16. $\sqrt[3]{}(c^3-3c^2a+3ca^2-a^3)\div\sqrt{}(b^2+c^2-2cb)$.

IV. *Change of the order of Terms. Like Terms.*

33. When all the terms of an expression are connected by the sign + it is indifferent in what *order* they are placed; thus $5+7$ and $7+5$ give the same result, namely, 12; and so also $a+b$ and $b+a$ give the same result, namely, the sum of the numbers which are represented by a and b. We may express this fact algebraically thus,

$$a+b=b+a.$$

Similarly, $$a+b+c=a+c+b=b+c+a.$$

34. When an expression consists of some terms preceded by the sign + and some terms preceded by the sign −, we may write the former terms first in any order we please, and the latter terms after them in any order we please. This is obvious from the common notions of arithmetic. Thus, for example,

$$7+8-2-3=8+7-2-3=7+8-3-2=8+7-3-2,$$
$$a+b-c-e=b+a-c-e=a+b-e-c=b+a-e-c.$$

35. In some cases we may change the order of the terms further, by mixing up the terms which are preceded by the sign − with those which are preceded by the sign +. Thus, for example, suppose that a represents 10, and b represents 6, and c represents 5, then

$$a+b-c=a-c+b=b-c+a;$$

for we arrive without any difficulty at 11 as the result in all the cases.

Suppose however that a represents 2, b represents 6, and c represents 5, then the expression $a-c+b$ presents a difficulty, because we are thus apparently required to take a greater number from a less, namely, 5 from 2. It will be convenient to agree that such an expression as $a-c+b$, when c is greater than a, shall be understood to mean the same thing as $a+b-c$. At present we shall not use such an expression as $a+b-c$ except when c is less than $a+b$;

so that $a+b-c$ will not cause any difficulty. Similarly, we shall consider $-b+a$ to mean the same thing as $a-b$.

36. Thus the numerical value of an expression remains the same, whatever may be the order of the terms which compose it. This, as we have seen, follows partly from our notions of addition and subtraction, and partly from an *agreement* as to the meaning which we ascribe to an expression when our ordinary arithmetical notions are not strictly applicable. Such an agreement is called in algebra a *convention*, and *conventional* is the corresponding adjective.

37. We shall often, as in Art. 34, have to distinguish the terms of an expression which are preceded by the sign + from the terms which are preceded by the sign −, and the following definition is accordingly adopted. The terms in an expression which are preceded by the sign + are called *positive* terms, and the terms which are preceded by the sign − are called *negative* terms. This definition is introduced merely for the sake of brevity, and no meaning is to be given to the words *positive* and *negative* beyond what is expressed in the definition.

38. It will be seen that a term may occur in an expression *preceded by no sign*, namely the first term. Such a term is *counted with the positive terms*, that is it is treated as if the sign + preceded it. It will be found that if such a change be made in the order of the terms, as to bring a term which originally stood first and was preceded by no sign, into any other place, then it will be preceded by the sign +. For example,

$$a+b-c=b+a-c=b-c+a;$$

here the term a has no sign before it in the first expression, but in the other equivalent expressions it is preceded by the sign +. Hence we have the following important addition to the definition in Art. 37; *if a term be preceded by no sign, the sign + is to be understood.*

39. Terms are said to be *like* when they do not differ at all, or differ only in their numerical coefficients; otherwise they are said to be *unlike*. Thus a, $4a$, and $7a$ are

like terms; a^2, $5a^2$, and $9a^2$ are like terms; a^2, ab, and b^2 are unlike terms.

40. An expression which contains like terms may be simplified. For example, consider the expression

$$6a - a + 3b + 5c - b + 3c - 2a;$$

by Art. 35 this expression is equivalent to

$$6a - a - 2a + 3b - b + 5c + 3c.$$

Now $6a - a - 2a = 3a$; for whatever number a may represent, if we subtract a from $6a$ we have $5a$ left, and then if we subtract $2a$ from $5a$ we have $3a$ left. Similarly $3b - b = 2b$; and $5c + 3c = 8c$. Thus the proposed expression may be put in the simpler form

$$3a + 2b + 8c.$$

Again; consider the expression $a - 3b - 4b$. This is equal to $a - 7b$. For if we have first to subtract $3b$ from a number a, and then to subtract $4b$ from the remainder, we shall obtain the required result in one operation by subtracting $7b$ from a; this follows from the common notions of Arithmetic. Thus

$$a - 3b - 4b = a - 7b.$$

41. There will be no difficulty now in giving a meaning to such a statement as the following,

$$-3b - 4b = -7b.$$

We cannot subtract $3b$ from nothing and then subtract $4b$ from the remainder, so that the statement just given is not here intelligible in itself, separated from the rest of an algebraical sentence in which it may occur, but it can be easily explained thus: if in the course of an algebraical operation we have to subtract $3b$ from a number and then to subtract $4b$ from the remainder, we may subtract $7b$ at once instead.

As the student advances in the subject he may be led to conjecture that it is possible to give some meaning to the proposed statement by itself, that is, apart from any other algebraical operation, and this conjecture will be found correct, when a larger treatise on Algebra can be

consulted with advantage; but the explanation which we have given will be sufficient for the present.

42. The simplifying of expressions by collecting like terms is the essential part of the processes of Addition and Subtraction in Algebra, as we shall see in the next two Chapters.

It may be useful for the beginner to notice that according to our definitions the following expressions are all equivalent to the single symbol a:

$$a^1,\ 1\times a,\ a\times 1,\ \frac{a}{1},$$

$$+a^1,\ +1\times a,\ +a\times 1,\ +\frac{a}{1}.$$

EXAMPLES. IV.

If $a=1$, $b=2$, $c=3$, $d=4$, $e=5$, find the numerical values of the following expressions:

1. $a-3b+4c$. 2. $a-b^2+c^3+d^3$.

3. $(a+b)(b+c)-(b+c)(c+d)+(c+d)(d+e)$.

4. $\dfrac{4a+3b}{b+c}-\dfrac{4c+3d}{b+d}+\dfrac{5d+4e}{a+d+e}$.

5. $(a-2b+3c)^2-(b-2c+3d)^2+(c-2d+3e)^2$.

6. $a^4-4a^3b+6a^2b^2-4ab^3+b^4$.

7. $\dfrac{b^2-2bc+c^2}{a^2-2ab+b^2}$. 8. $\dfrac{a^4-4a^3c+6a^2c^2-4ac^3+c^4}{b^4-4b^3c+6b^2c^2-4bc^3+c^4}$.

9. $7a-2b-3c-4a+5b+4c+2a$.

10. $5a^2+3ab-2b^2-ab+9b^2-2ab-7b^2$.

11. $3a^3-2a^2+5a+a^3+a+9a^2-4a^3-6a$.

12. $\dfrac{a^2+2ab+b^2}{a+b}-\dfrac{b^2+2bc+c^2}{b+c}+\dfrac{c^2+2cd+d^2}{c+d}$.

13. $\sqrt{(4c^2+5d^2+e)}$. 14. $\sqrt{(e^2+d^2+c^2-a^2)}$.

15. $\sqrt[3]{(2e^2+7b)}$. 16. $\sqrt[4]{(2b^2+c^2-a)}$.

V. *Addition.*

43. It is convenient to make three cases in Addition, namely, I. When the terms are all like terms and have the same sign; II. When the terms are all like terms but have not all the same sign; III. When the terms are not all like terms. We shall take these three cases in order.

44. I. To add like terms which have the same sign. *Add the numerical coefficients, prefix the common sign, and annex the common letters.*

For example, $6a+3a+7a=16a$,

$$-2bc-7bc-9bc=-18bc.$$

In the first example $6a$ is equivalent to $+6a$, and $16a$ to $+16a$. See Art. 38.

45. II. To add like terms which have not all the same sign. *Add all the positive numerical coefficients into one sum, and all the negative numerical coefficients into another; take the difference of these two sums, prefix the sign of the greater, and annex the common letters.*

For example,

$$7a-3a+11a+a-5a-2a=19a-10a=9a,$$

$$2bc-7bc-3bc+4bc+5bc-6bc=11bc-16bc=-5bc.$$

46. III. To add terms which are not all like terms. *Add together the terms which are like terms by the rule in the second case, and put down the other terms each preceded by its proper sign.*

For example; add together

$$4a+5b-7c+3d,\ 3a-b+2c+5d,\ 9a-2b-c-d,$$
$$\text{and } -a+3b+4c-3d+e.$$

It is convenient to arrange the terms in columns, so that like terms shall stand in the same column; thus we have

$$
\begin{array}{l}
4a+5b-7c+3d \\
3a-\ \ b+2c+5d \\
9a-2b-\ \ c-\ \ d \\
-a+3b+4c-3d+e \\
\hline
15a+5b-2c+4d+e
\end{array}
$$

Here the terms $4a$, $3a$, $9a$, and $-a$ are all like terms; the sum of the positive coefficients is 16; there is one term with a negative coefficient, namely $-a$, of which the coefficient is 1. The difference of 16 and 1 is 15; so that we obtain $+15a$ from these like terms; the sign $+$ may however be omitted by Art. 38. Similarly we have $5b-b-2b+3b=5b$. And so on.

47. In the following examples the terms are arranged suitably in columns:

$$
\begin{array}{rrrr}
x^3 & +2x^2 & -3x & +1 \\
4x^3 & +7x^2 & +x & -9 \\
-2x^3 & +x^2 & -9x & +8 \\
-3x^3 & -x^2 & +10x & -1 \\
\hline
 & 9x^2 & -x & -1
\end{array}
\qquad\qquad
\begin{array}{rrrr}
a^2 & +ab & +b^2 & -c \\
3a^2 & -3ab & -7b^2 & \\
4a^2 & +5ab & +9b^2 & \\
a^2 & -3ab & -3b^2 & \\
\hline
9a^2 & & & -c
\end{array}
$$

In the first example we have in the first column $x^3+4x^3-2x^3-3x^3$, that is $5x^3-5x^3$, that is, nothing; this is usually expressed by saying *the terms which involve* x^3 *cancel each other.*

Similarly, in the second example, the terms which involve ab cancel each other; and so also do the terms which involve b^2.

$$
\begin{array}{rrrrr}
7x^2 & -3xy & & +x & \\
3x^2 & & -y^2 & +3x & -y \\
-2x^2 & +4xy & +5y^2 & -x & -2y \\
 & -7xy & -y^2 & +9x & -5y \\
4x^2 & & +4y^2 & -2x & \\
\hline
12x^2 & -6xy & +7y^2 & +10x & -8y
\end{array}
$$

EXAMPLES. V.

Add together

1. $3a-2b,\quad 4a-5b,\quad 7a-11b,\quad a+9b.$
2. $4x^2-3y^2,\quad 2x^2-5y^2,\quad -x^2+y^2,\quad -2x^2+4y^2.$
3. $5a+3b+c,\quad 3a+3b+3c,\quad a+3b+5c.$
4. $3x+2y-z,\quad 2x-2y+2z,\quad -x+2y+3z.$
5. $7a-4b+c,\quad 6a+3b-5c,\quad -12a+4c.$
6. $x-4a+b,\quad 3x+2b,\quad a-x-5b.$
7. $a+b-c,\quad b+c-a,\quad c+a-b,\quad a+b-c.$
8. $a+2b+3c,\quad 2a-b-2c,\quad b-a-c,\quad c-a-b.$
9. $a-2b+3c-4d,\quad 3b-4c+5d-2a,\quad 5c-6d+3a-4b,$ $7d-4a+5b-4c.$
10. $x^3-4x^2+5x-3,\quad 2x^3-7x^2-14x+5,\quad -x^3+9x^2+x+8.$
11. $x^4-2x^3+3x^2,\quad x^3+x^2+x,\quad 4x^4+5x^3,\quad 2x^2+3x-4,$ $-3x^2-2x-5.$
12. $a^3-3a^2b+3ab^2-b^3,\quad 2a^3+5a^2b-6ab^2-7b^2,$ $a^3-ab^2+2b^3.$
13. $x^3-2ax^2+a^2x+a^3,\quad x^3+3ax^2,\quad 2a^3-ax^2-2x^3.$
14. $2ab-3ax^2+2a^2x,\quad 12ab+10ax^2-6a^2x,$ $-8ab+ax^3-5a^2x.$
15. $x^2+y^4+z^3,\quad -4x^2-5z^3,\quad 8x^2-7y^4+10z^3,\quad 6y^4-6z^3.$
16. $3x^2-4xy+y^2+2x+3y-7,\quad 2x^2-4y^2+3x-5y+8,$ $10xy+8y^2+9y,\quad 5x^2-6xy+3y^2+7x-7y+11.$
17. $x^4-4x^3y+6x^2y^2-4xy^3+y^4,\quad 4x^3y-12x^2y^2+12xy^3-4y^4,$ $6x^2y^2-12xy^3+6y^4,\quad 4xy^3-4y^4,\quad y^4.$
18. $x^3+xy^2+xz^2-x^2y-xyz-x^2z,$ $x^2y+y^3+yz^2-xy^2-y^2z-xyz,$ $x^2z+y^2z+z^3-xyz-yz^2-xz^2.$

VI. *Subtraction.*

48. Suppose we have to take $7+3$ from 12; the result is the same as if we first take 7 from 12, and then take 3 from the remainder; that is, the result is denoted by $12-7-3$.

Thus $$12-(7+3)=12-7-3.$$

Here we enclose $7+3$ in brackets in the first expression, because we are to take the *whole* of $7+3$ from 12; see Art. 29.

Similarly $$20-(5+4+2)=20-5-4-2.$$

In like manner, suppose we have to take $b+c$ from a; the result is the same as if we first take b from a, and then take c from the remainder; that is, the result is denoted by $a-b-c$.

Thus $$a-(b+c)=a-b-c.$$

Here we enclose $b+c$ in brackets in the first expression, because we are to take the whole of $b+c$ from a.

Similarly $$a-(b+c+d)=a-b-c-d.$$

49. Next suppose we have to take $7-3$ from 12. If we take 7 from 12 we obtain $12-7$; but we have thus taken too much from 12, for we had to take, not 7, but 7 diminished by 3. Hence we must increase the result by 3; and thus we obtain $12-(7-3)=12-7+3$.

Similarly $$12-(7+3-2)=12-7-3+2.$$

In like manner, suppose we have to take $b-c$ from a. If we take b from a we obtain $a-b$; but we have thus taken too much from a, for we had to take, not b, but b diminished by c. Hence we must increase the result by c; and thus we obtain $a-(b-c)=a-b+c$.

Similarly $$a-(b+c-d)=a-b-c+d.$$

50. Consider the example

$$a-(b+c-d)=a-b-c+d;$$

that is, if $b+c-d$ be subtracted from a the result is

$a-b-c+d$. Here we see that, in the expression to be subtracted there is a term $-d$, and in the result there is the corresponding term $+d$; also in the expression to be subtracted there is a term $+c$, and in the result there is a term $-c$; also in the expression to be subtracted there is a term b, and in the result there is a term $-b$.

From considering this example, and the others in the two preceding Articles we obtain the following rule for Subtraction: *change the signs of all the terms in the expression to be subtracted, and then collect the terms as in Addition.*

For example; from $4x-3y+2z$ subtract $3x-y+z$. Change the signs of all the terms to be subtracted; thus we obtain $-3x+y-z$; then collect as in addition; thus

$$4x-3y+2z-3x+y-z=x-2y+z.$$

From $3x^4+5x^3-6x^2-7x+5$ take $2x^4-2x^3+5x^2-6x-7$.

Change the signs of all the terms to be subtracted and proceed as in addition; thus we have

$$\begin{array}{r} 3x^4+5x^3-6x^2-7x+5 \\ -2x^4+2x^3-5x^2+6x+7 \\ \hline x^4+7x^3-11x^2-x+12 \end{array}$$

The beginner will find it prudent at first to go through the operation as fully as we have done here; but he may gradually accustom himself to putting down the result without actually changing all the signs, but merely supposing it done.

51. We have seen that

$$a-(b-c)=a-b+c.$$

Thus corresponding to the term $-c$ in the expression to be subtracted we have $+c$ in the result. Hence it is not uncommon to find such an example as the following proposed for exercise: from a subtract $-c$; and the result required is $a+c$. The beginner may explain this in the manner of Art. 41, by considering it as having a meaning, not in itself, but in connexion with some other parts of an algebraical operation.

It is usual however to offer some remarks which will serve to impress results on the attention of the beginner, and perhaps at the same time to suggest reasons for them.

Thus we may say that $a = a + c - c$, so that if we subtract $-c$ from a there remains $a + c$.

Or we may say that + and − denote operations the reverse of each other; thus $-c$ denotes the reverse of $+c$, and so $-(-c)$ will denote the reverse of the reverse of $+c$, that is, $-(-c)$ is equivalent to $+c$.

But, as we have implied in Art. 41, the beginner must be content to defer until a later period the complete explanation of the meaning of operations performed on *negative quantities*, that is, on quantities denoted by letters with the sign − prefixed.

It should be observed that the words *addition* and *subtraction* are not used in quite the same sense in Algebra as in Arithmetic. In Arithmetic addition always produces *increase* and subtraction *decrease;* but in Algebra we may speak of adding −3 to 5, and obtaining the *Algebraical sum* 2; or we may speak of subtracting −3 from 5, and obtaining the *Algebraical remainder* 8.

EXAMPLES. VI.

1. From $7a + 14b$ subtract $4a + 10b$.

2. From $6a - 2b - c$ subtract $2a - 2b - 3c$.

3. From $3a - 2b + 3c$ subtract $2a - 7b - c - d$.

4. From $7x^2 - 8x - 1$ subtract $5x^2 - 6x + 3$.

5. From $4x^4 - 3x^3 - 2x^2 - 7x + 9$
   subtract $x^4 - 2x^3 - 2x^2 + 7x - 9$.

6. From $2x^2 - 2ax + 3a^2$ subtract $x^2 - ax + a^2$.

7. From $x^2 - 3xy - y^2 + yz - 2z^2$
   subtract $x^2 + 2xy + 5xz - 3y^2 - 2z^2$.

8. From $5x^2 + 6xy - 12xz - 4y^2 - 7yz - 5z^2$
   subtract $2x^2 - 7xy + 4xz - 3y^2 + 6yz - 5z^2$.

9. From $a^3 - 3a^2b + 3ab^2 - b^3$ subtract $-a^3 + 3a^2b - 3ab^2 + b^3$.

10. From $7x^3 - 2x^2 + 2x + 2$ subtract $4x^3 - 2x^2 - 2x - 14$,
   and from the remainder subtract $2x^3 - 8x^2 + 4x + 16$.

VII. *Brackets.*

52. On account of the extensive use which is made of brackets in Algebra, it is necessary that the student should observe very carefully the rules respecting them, and we shall state them here distinctly.

When an expression within a pair of brackets is preceded by the sign + the brackets may be removed.

When an expression within a pair of brackets is preceded by the sign – the brackets may be removed if the sign of every term within the brackets be changed.

Thus, for example,

$$a-b+(c-d+e)=a-b+c-d+e,$$
$$a-b-(c-d+e)=a-b-c+d-e.$$

The second rule has already been illustrated in Art. 50; it is in fact the *rule for Subtraction*. The first rule might be illustrated in a similar manner.

53. In particular the student must notice such statements as the following:

$$+(-d)=-d, \quad -(-d)=+d, \quad +(+e)=+e, \quad -(+e)=-e.$$

These must be assumed as rules by the student, which he may to some extent explain, as in Art. 41.

54. Expressions may occur with more than one pair of brackets; these brackets may be removed in succession by the preceding rules *beginning with the inside pair*. Thus, for example,

$$a+\{b+(c-d)\}=a+\{b+c-d\}=a+b+c-d,$$
$$a+\{b-(c-d)\}=a+\{b-c+d\}=a+b-c+d,$$
$$a-\{b+(c-d)\}=a-\{b+c-d\}=a-b-c+d,$$
$$a-\{b-(c-d)\}=a-\{b-c+d\}=a-b+c-d.$$

Similarly,

$$a-[b-\{c-(d-e)\}]=a-[b-\{c-d+e\}]$$
$$=a-[b-c+d-e]=a-b+c-d+e.$$

It will be seen in these examples that, to prevent confusion between various pairs of brackets, we use brackets

of different *shapes;* we might distinguish by using brackets of the same shape but of different *sizes*.

A vinculum is equivalent to a bracket; see Art. 30. Thus, for example,

$$a-[b-\{c-(d-\overline{e-f})\}]=a-[b-\{c-(d-e+f)\}]$$
$$=a-[b-\{c-d+e-f\}]=a-[b-c+d-e+f]$$
$$=a-b+c-d+e-f.$$

55. The beginner is recommended always to remove brackets in the order shewn in the preceding Article; namely, by removing first the innermost pair, next the innermost pair of all which remain, and so on. We may however vary the order; but if we remove a pair of brackets including another bracketed expression within it, we must *make no change in the signs of the included expression.* In fact such an included expression counts as a single term. Thus, for example,

$$a+\{b+(c-d)\}=a+b+(c-d)=a+b+c-d,$$
$$a+\{b-(c-d)\}=a+b-(c-d)=a+b-c+d,$$
$$a-\{b+(c-d)\}=a-b-(c-d)=a-b-c+d,$$
$$a-\{b-(c-d)\}=a-b+(c-d)=a-b+c-d.$$

Also, $a-[b-\{c-(d-e)\}]=a-b+\{c-(d-e)\}$
$$=a-b+c-(d-e)=a-b+c-d+e.$$

And in like manner, $a-[b-\{c-(d-\overline{e-f})\}]$
$$=a-b+\{c-(d-\overline{e-f})\}=a-b+c-(d-\overline{e-f})$$
$$=a-b+c-d+\overline{e-f}=a-b+c-d+e-f.$$

56. It is often convenient to put two or more terms within brackets; the rules for introducing brackets follow immediately from those for removing brackets.

Any number of terms in an expression may be put within a pair of brackets and the sign + placed before the whole.

Any number of terms in an expression may be put within a pair of brackets and the sign − placed before the whole, provided the sign of every term within the brackets be changed.

Thus, for example, $a-b+c-d+e$

$=a-b+(c-d+e)$, or $=a-b+c+(-d+e)$,

or $=a-(b-c+d-e)$, or $=a-b-(-c+d-e)$.

In like manner more than one pair of brackets may be introduced. Thus, for example,

$$a-b+c-d+e=a-\{b-c+d-e\}=a-\{b-(c-d+e)\}.$$

Examples. VII.

Simplify the following expressions by removing the brackets and collecting like terms:

1. $3a-b-(2a-b)$.
2. $a-b+c-(a-b-c)$.
3. $1-(1-a)+(1-a+a^2)-(1-a+a^2-a^3)$.
4. $a+b+(7a-b)-(2a-3b)-(5a+6b)$.
5. $a-b+c-(b-a+c)+(c-a+b)-(a-c+b)$.
6. $2x-3y-3z-(x-y+2z)+(x+4y+5z)-(z-x-y)$.
7. $a-\{b-c-(d-e)\}$.
8. $2a-(2b-d)-\{a-b-(2c-2d)\}$.
9. $a-\{2b-(3c+2b-a)\}$.
10. $2a-\{b-(a-2b)\}$.
11. $3a-\{b+(2a-b)-(a-b)\}$.
12. $7a-[3a-\{4a-(5a-2a)\}]$.
13. $3a-[b-\{a+(b-3a)\}]$.
14. $6a-[4b-\{4a-(6a-4b)\}]$.
15. $2a-(3b+2c)-[5b-(6c-6b)+5c-\{2a-(c+2b)\}]$.
16. $a-[2b+\{3c-3a-(a+b)\}+\{2a-(b+c)\}]$.
17. $16-\{5-2x-[1-(3-x)]\}$.
18. $15x-\{4-[3-5x-(3x-7)]\}$.
19. $2a-[2a-\{2a-(2a-\overline{2a-a})\}]$.
20. $16-x-[7x-\{8x-(9x-\overline{3x-6x})\}]$.
21. $2x-[3y-\{4x-(5y-\overline{6x-7y})\}]$.
22. $2a-[3b+(2b-c)-4c+\{2a-(3b-\overline{c-2b})\}]$.
23. $a-[5b-\{a-(5c-\overline{2c-b}-4b)+2a-(a-\overline{2b+c})\}]$.
24. $x^4-[4x^3-\{6x^2-(4x-1)\}]-(x^4+4x^3+6x^2+4x+1)$.

VIII. *Multiplication.*

57. The student is supposed to know that the product of any number of factors is the same in whatever order the factors may be taken; thus $2 \times 3 \times 5 = 2 \times 5 \times 3 = 3 \times 5 \times 2$; and so on. In like manner $abc = acb = bca$, and so on.

Thus also $c(a+b)$ and $(a+b)c$ are equal, for each denotes the product of the same two factors; one factor being c, and the other factor $a+b$.

It is convenient to make three cases in Multiplication, namely, I. The multiplication of simple expressions; II. The multiplication of a compound expression by a simple expression; III. The multiplication of compound expressions. We shall take these three cases in order.

58. I. Suppose we have to multiply $3a$ by $4b$. The product may be written at full thus $3 \times a \times 4 \times b$, or thus $3 \times 4 \times a \times b$; and it is therefore equal to $12ab$. Hence we have the following rule for the multiplication of simple expressions; *multiply together the numerical coefficients and put the letters after this product.*

Thus for example,

$$7a \times 3bc = 21abc,$$

$$4a \times 5b \times 3c = 60abc.$$

59. *The powers of the same number are multiplied together by adding the exponents.*

For example, suppose we have to multiply a^3 by a^2.

By Art. 16, $\quad a^3 = a \times a \times a,$

and $\quad a^2 = a \times a;$

therefore $\quad a^3 \times a^2 = a \times a \times a \times a \times a = a^5 = a^{3+2}.$

Similarly, $\quad c^4 \times c^3 = c \times c \times c \times c \times c \times c \times c = c^7 = c^{4+3}.$

In like manner the rule may be seen to be true in any other case.

60. II. Suppose we have to multiply $a+b$ by 3. We have

$$3(a+b)=a+b+a+b+a+b=3a+3b.$$

Similarly, $7(a+b)=7a+7b.$

In the same manner suppose we have to multiply $a+b$ by c. We have

$$c(a+b)=ca+cb.$$

In the same manner we have

$$3(a-b)=3a-3b, \quad 7(a-b)=7a-7b, \quad c(a-b)=ca-cb.$$

Thus we have the following rule for the multiplication of a compound expression by a simple expression; *multiply each term of the compound expression by the simple expression, and put the sign of the term before the result; and collect these results to form the complete product.*

61. III. Suppose we have to multiply $a+b$ by $c+d$.

As in the second case we have

$$(a+b)\,(c+d)=a(c+d)+b(c+d);$$

also $a(c+d)=ac+ad, \qquad b(c+d)=bc+bd;$

therefore $(a+b)(c+d)=ac+ad+bc+bd.$

Again; multiply $a-b$ by $c+d$.

$$(a-b)(c+d)=a(c+d)-b(c+d);$$

also $a(c+d)=ac+ad, \quad b(c+d)=bc+bd;$

therefore

$$(a-b)(c+d)=ac+ad-(bc+bd)=ac+ad-bc-bd.$$

Similarly; multiply $a+b$ by $c-d$.

$$(a+b)(c-d)=(c-d)(a+b)=c(a+b)-d(a+b)$$
$$=ca+cb-(da+db)=ca+cb-da-db.$$

Lastly; multiply $a-b$ by $c-d$.

$$(a-b)(c-d)=(c-d)a-(c-d)b;$$

also $(c-d)a=ac-ad, \quad (c-d)b=bc-bd;$

therefore

$$(a-b)(c-d)=ac-ad-(bc-bd)=ac-ad-bc+bd.$$

Let us now consider the last result. By Art. 38 we may write it thus,

$$(+a-b)(+c-d)=+ac-ad-bc+bd.$$

We see that corresponding to the $+a$ which occurs in the multiplicand and the $+c$ which occurs in the multiplier there is a term $+ac$ in the product; corresponding to the terms $+a$ and $-d$ there is a term $-ad$ in the product; corresponding to the terms $-b$ and $+c$ there is a term $-bc$ in the product; and corresponding to the terms $-b$ and $-d$ there is a term $+bd$ in the product.

Similar observations may be made respecting the other three results; and these observations are briefly collected in the following important rule in multiplication: *like signs produce + and unlike signs −.* This rule is called the *Rule of Signs*, and we shall often refer to it by this name.

62. We can now give the general rule for multiplying algebraical expressions; *multiply each term of the multiplicand by each term of the multiplier; if the terms have the same sign prefix the sign + to the product, if they have different signs prefix the sign −; then collect these results to form the complete product.*

For example; multiply $2a+3b-4c$ by $3a-4b$. Here

$$\begin{aligned}(2a+3b-4c)(3a-4b) &= 3a(2a+3b-4c)-4b(2a+3b-4c)\\ &= 6a^2+9ab-12ac-(8ab+12b^2-16bc)\\ &= 6a^2+9ab-12ac-8ab-12b^2+16bc.\end{aligned}$$

This is the result which the rule will give; we may simplify the result and reduce it to

$$6a^2+ab-12ac-12b^2+16bc.$$

We might illustrate the rule by using it to multiply $6-3+2$ by $7+3-4$; it will be found that on working by the rule, and collecting the terms, the result is 30, that is 5×6, as it should be.

63. The student will sometimes find such examples as the following proposed: multiply $2a$ by $-4b$, or multiply $-4c$ by $3a$, or multiply $-4c$ by $-4b$.

The results which are required are the following,

$$\begin{aligned}2a \times -4b &= -\ 8ab,\\ -4c \times\ \ 3a &= -12ac,\\ -4c \times -4b &= \ \ 16bc.\end{aligned}$$

The student may attach a meaning to these operations in the manner we have already explained; see Article 41.

Thus the statement $-4c \times -4b = 16bc$ may be understood to mean, that if $-4c$ occur among the terms of a multiplicand and $-4b$ occur among the terms of a multiplier, there will be a term $16bc$ in the product corresponding to them.

Particular cases of these examples are

$$2a \times -4 = -8a, \qquad 2 \times -4 = -8, \qquad 2 \times -1 = -2.$$

64. Since then such examples may be given as those in the preceding Article, it becomes necessary to take account of them in our rules; and accordingly the rules for multiplication may be conveniently presented thus:

To multiply simple terms; *multiply together the numerical coefficients, put the letters after this product and determine the sign by the Rule of Signs.*

To multiply expressions; *multiply each term in one expression by each term in the other by the rule for multiplying simple terms, and collect these partial products to form the complete product.*

65. We shall now give some examples of multiplication arranged in a convenient form.

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \quad +ab+b^2 \\ \hline a^2+2ab+b^2 \end{array} \qquad \begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline a^2 \qquad -b^2 \end{array} \qquad \begin{array}{l} x^2+3x \\ x-1 \\ \hline x^3+3x^2 \\ \quad -x^2-3x \\ \hline x^3+2x^2-3x \end{array}$$

$$\begin{array}{l} a^2-ab+b^2 \\ a+b \\ \hline a^3-a^2b+ab^2 \\ \quad +a^2b-ab^2+b^3 \\ \hline a^3 \qquad\qquad +b^3 \end{array} \qquad \begin{array}{rrrrr} 3a^2- & 4ab+ & 5b^2 & & \\ a^2- & 2ab+ & 3b^2 & & \\ \hline 3a^4- & 4a^3b+ & 5a^2b^2 & & \\ & -6a^3b+ & 8a^2b^2 & -10ab^3 & \\ & & +9a^2b^2 & -12ab^3 & +15b^4 \\ \hline 3a^4 & -10a^3b & +22a^2b^2 & -22ab^3 & +15b^4 \end{array}$$

Consider the last example. We take the first term in the multiplier, namely a^2, and multiply all the terms in the multiplicand by it, paying attention to the *Rule of Signs;* thus we obtain $3a^4-4a^3b+5a^2b^2$. We take next the second term of the multiplier, namely $-2ab$, and multiply all the terms in the multiplicand by it, paying attention to the *Rule of Signs;* thus we obtain $-6a^3b+8a^2b^2-10ab^3$. Then we take the last term of the multiplier, namely $3b^2$, and multiply all the terms in the multiplicand by it, paying attention to the *Rule of Signs;* thus we obtain $+9a^2b^2-12ab^3+15b^4$.

We arrange the terms which we thus obtain, so that *like terms may stand in the same column;* this is a very useful arrangement, because it enables us to collect the terms easily and safely, in order to obtain the final result. In the present example the final result is

$$3a^4-10a^3b+22a^2b^2-22ab^3+15b^4.$$

66. The student should observe that with the view of bringing like terms of the product into the same column the terms of the multiplicand and multiplier are arranged in a certain order. We fix on some letter which occurs in many of the terms and arrange the terms *according to the powers of that letter.* Thus, taking the last example, we fix on the letter a; we put first in the multiplicand the term $3a^2$, which contains the highest power of a, namely the second power; next we put the term $-4ab$ which contains the next power of a, namely the first power; and last we put the term $5b^2$, which does not contain a at all. The multiplicand is then said to be arranged *according to descending powers of* a. We arrange the multiplier in the same way.

We might also have arranged both multiplicand and multiplier in reverse order, in which case they would be arranged *according to ascending powers of* a. It is of no consequence which order we adopt, but we must take the *same* order for the multiplicand and the multiplier.

67. We shall now give some more examples.

Multiply $1+2x-3x^2+x^4$ by x^3-2x-2. Arrange according to descending powers of x.

$$
\begin{array}{l}
x^4-3x^2+2x\ +1 \\
x^3-2x\ -2 \\
\hline
x^7-3x^5+2x^4+x^3 \\
\quad -2x^5 \qquad +6x^3-4x^2-2x \\
\qquad\quad -2x^4 \qquad +6x^2-4x-2 \\
\hline
x^7-5x^5 \qquad +7x^3+2x^2-6x-2
\end{array}
$$

Multiply $a^2+b^2+c^2-ab-bc-ca$ by $a+b+c$.

Arrange according to descending powers of a.

$$
\begin{array}{l}
a^2-ab-ac+b^2-bc+c^2 \\
a\ +\ b\ +\ c \\
\hline
a^3-a^2b-a^2c+ab^2-abc+ac^2 \\
\quad +a^2b \qquad -ab^2-abc \qquad +b^3-b^2c+bc^2 \\
\qquad\quad +a^2c \qquad -abc-ac^2 \qquad +b^2c-bc^2+c^3 \\
\hline
a^3 \qquad\qquad -3abc \qquad +b^3 \qquad\qquad +c^3
\end{array}
$$

This example might also be worked with the aid of brackets, thus,

$$
\begin{array}{l}
a^2-a(b+c)+b^2-bc+c^2 \\
a\ +\ (b+c) \\
\hline
a^3-a^2(b+c)+a(b^2-bc+c^2) \\
\quad +a^2(b+c)-a(b+c)(b+c)+(b+c)(b^2-bc+c^2) \\
\hline
\end{array}
$$

Then we have $a(b^2-bc+c^2)-a(b+c)(b+c)$

$$=a\{b^2-bc+c^2-(b+c)(b+c)\}$$

$$=a\{b^2-bc+c^2-(b^2+2bc+c^2)\}$$

$$=a\{b^2-bc+c^2-b^2-2bc-c^2\}=-3abc;$$

and $$(b+c)(b^2-bc+c^2)=b^3+c^3.$$

Thus, as before, the result is $a^3+b^3+c^3-3abc$.

Multiply together $x-a$, $x-b$, $x-c$.

$$\begin{array}{l}
x\;-a\\
x\;-b\\
\hline
x^2-ax\\
\quad\;\;-bx+ab\\
\hline
x^2-(a+b)x+ab\\
x\;-c\\
\hline
x^3-(a+b)x^2+abx\\
\qquad\qquad\;-cx^2+(a+b)cx-abc\\
\hline
x^3-(a+b+c)x^2+(ab+ac+bc)x-abc
\end{array}$$

The student should notice that he can make two exercises in multiplication from every example in which the multiplicand and multiplier are different compound expressions, by changing the original multiplier into the multiplicand, and the original multiplicand into multiplier. The result obtained should be the same, which will be a test of the correctness of his work.

EXAMPLES. VIII.

Multiply

1. $2x^3$ by $4x^2$. 2. $3a^4$ by $4a^5$. 3. $2a^2b$ by $2ab^2$.

4. $3x^3y^2z$ by $5x^4y^3z^2$. 5. $7x^4y^2$ by $7y^2z^4$.

6. $4a^2-3b$ by $3ab$. 7. $8a^2-9ab$ by $3a^2$.

8. $3x^2-4y^2+5z^2$ by $2x^2y$.

9. $x^2y^3-y^3z^4+z^4x^2$ by $x^2y^2z^2$.

10. $2xy^2z^3+3x^2y^3z-5x^3yz^2$ by $2xy^2z$.

11. $2x-y$ by $2y+x$.

12. $2x^3+4x^2+8x+16$ by $3x-6$.

13. x^3+x^2+x-1 by $x-1$.

14. $1+4x-10x^2$ by $1-6x+3x^2$.

15. $x^3-4x^2+11x-24$ by x^2+4x+5.

16. $x^3+4x^2+5x-24$ by $x^2-4x+11$.

17. x^3-7x^2+5x+1 by $2x^2-4x+1$.

18. $x^3+6x^2+24x+60$ by $x^3-6x^2+12x+12$.

19. x^3-2x^2+3x-4 by $4x^3+3x^2+2x+1$.

20. $x^4-2x^3+3x^2-2x+1$ by $x^4+2x^3+3x^2+2x+1$.

21. x^2-3ax by $x+3a$.

22. $a^2+2ax-x^2$ by $a^2+2ax+x^2$.

23. $2b^2+3ab-a^2$ by $7a-5b$.

24. a^2-ab+b^2 by a^2+ab-b^2.

25. $a^2-ab+2b^2$ by $a^2+ab+2b^2$.

26. $4x^2-3xy-y^2$ by $3x-2y$.

27. $x^5-x^4y+xy^4-y^5$ by $x+y$.

28. $2x^2+3xy+4y^2$ by $3x^2+4xy+y^2$.

29. $x^2+y^2-xy+x+y-1$ by $x+y-1$.

30. $x^4+2x^3y+4x^2y^2+8xy^3+16y^4$ by $x-2y$.

31. $81x^4+27x^3y+9x^2y^2+3xy^3+y^4$ by $3x-y$.

32. $x+2y-3z$ by $x-2y+3z$.

33. $a^2-ax+bx+b^2$ by $a+b+x$.

34. $a^2+b^2+c^2-bc-ca-ab$ by $a+b+c$.

35. $a^2+4bx+4b^2x^2$ by $a^2-4bx+4b^2x^2$.

36. $a^2-2ab+b^2+c^2$ by $a^2+2ab+b^2-c^2$.

Multiply the following expressions together

37. $x-a$, $x+a$, x^2+a^2.

38. $x+a$, $x+b$, $x+c$.

39. x^2-ax+a^2, x^2+ax+a^2, $x^4-a^2x^2+a^4$.

40. $x-2a$, $x-a$, $x+a$, $x+2a$.

IX. *Division.*

68. Division, as in Arithmetic, is the inverse of Multiplication. In Multiplication we determine the product arising from two given factors; in Division we have given the product and one of the factors, and we have to determine the other factor. The factor to be determined is called the *quotient.*

The present section therefore is closely connected with the preceding section, as we have now in fact to undo the operations there performed. It is convenient to make three cases in Division, namely, I. The division of one simple expression by another; II. The division of a compound expression by a simple expression; III. The division of one compound expression by another.

69. I. We have already shewn in Art. 10 how to denote that one expression is to be divided by another. For example, if $5a$ is to be divided by $2c$ the quotient is indicated thus: $5a \div 2c$, or more usually $\frac{5a}{2c}$.

It may happen that some of the factors of the divisor occur in the dividend; in this case the expression for the quotient can be simplified by a principle already used in Arithmetic. Suppose, for example, that $15a^2b$ is to be divided by $6bc$; then the quotient is denoted by $\frac{15a^2b}{6bc}$. Here the dividend $15a^2b = 5a^2 \times 3b$; and the divisor $6bc = 2c \times 3b$; thus the factor $3b$ occurs in both dividend and divisor. Then, as in Arithmetic, we may remove this common factor, and denote the quotient by $\frac{5a^2}{2c}$; thus $\frac{15a^2b}{6bc} = \frac{5a^2}{2c}$.

It may happen that all the factors which occur in the divisor may be removed in this manner. Thus suppose, for example, that $24abx$ is to be divided by $8ax$:

$$\frac{24abx}{8ax}=\frac{3b\times 8ax}{8ax}=3b.$$

70. The rule with respect to the *sign* of the quotient may be obtained from an examination of the cases which occur in Multiplication.

For example, we have

$$4ab\times 3c=12abc;$$

therefore $$\frac{12abc}{4ab}=3c,\qquad \frac{12abc}{3c}=4ab.$$

$$4ab\times -3c=-12abc;$$

therefore $$\frac{-12abc}{4ab}=-3c,\qquad \frac{-12abc}{-3c}=4ab.$$

$$-4ab\times 3c=-12abc;$$

therefore $$\frac{-12abc}{-4ab}=3c,\qquad \frac{-12abc}{3c}=-4ab.$$

$$-4ab\times -3c=12abc;$$

therefore $$\frac{12abc}{-4ab}=-3c,\qquad \frac{12abc}{-3c}=-4ab.$$

Thus it will be seen that the *Rule of Signs* holds in Division as well as in Multiplication.

71. Hence we have the following rule for dividing one simple expression by another: *Write the dividend over the divisor with a line between them; if the expressions have common factors, remove the common factors; prefix the sign + if the expressions have the same sign and the sign − if they have different signs.*

72. *One power of any number is divided by another power of the same number, by subtracting the index of the latter power from the index of the former.*

For example, suppose we have to divide a^5 by a^3.

By Art. 16, $a^5 = a \times a \times a \times a \times a,$

$a^3 = a \times a \times a;$

therefore $$\frac{a^5}{a^3} = \frac{a \times a \times a \times a \times a}{a \times a \times a} = a \times a = a^2 = a^{5-3}.$$

Similarly $$\frac{c^7}{c^4} = \frac{c \times c \times c \times c \times c \times c \times c}{c \times c \times c \times c} = c \times c \times c = c^3 = c^{7-4}.$$

In like manner the rule may be shewn to be true in any other case.

Or we may shew the truth of the rule thus:

by Art. 59, $c^4 \times c^3 = c^7,$

therefore $$\frac{c^7}{c^4} = c^3, \qquad \frac{c^7}{c^3} = c^4.$$

73. If any power of a number occurs in the dividend and a higher power of the same number in the divisor, the quotient can be simplified by Arts. 71, and 72. Suppose, for example, that $4ab^2$ is to be divided by $3cb^5$; then the quotient is denoted by $\frac{4ab^2}{3cb^5}$. The factor b^2 occurs in both dividend and divisor; this may be removed, and the quotient denoted by $\frac{4a}{3cb^3}$; thus $\frac{4ab^2}{3cb^5} = \frac{4a}{3cb^3}$.

74. II. The rule for dividing a compound expression by a simple expression will be obtained from an examination of the corresponding case in Multiplication.

For example, we have

$$(a - b)\,c = ac - bc;$$

therefore $$\frac{ac - bc}{c} = a - b.$$

$$(a - b) \times -c = -ac + bc;$$

therefore $$\frac{-ac + bc}{-c} = a - b.$$

Hence we have the following rule for dividing a compound expression by a simple expression: *divide each term of the dividend by the divisor, by the rule in the first case, and collect the results to form the complete quotient.*

For example, $\dfrac{4a^3 - 3abc + a^2c}{a} = 4a^2 - 3bc + ac.$

75. III. To divide one compound expression by another we must proceed as in the operation called Long Division in Arithmetic. The following rule may be given. *Arrange both dividend and divisor according to ascending powers of some common letter, or both according to descending powers of some common letter. Divide the first term of the dividend by the first term of the divisor, and put the result for the first term of the quotient; multiply the whole divisor by this term and subtract the product from the dividend. To the remainder join as many terms of the dividend, taken in order, as may be required, and repeat the whole operation. Continue the process until all the terms of the dividend have been taken down.*

The reason for this rule is the same as that for the rule of Long Division in Arithmetic, namely, that we may break the dividend up into parts and find how often the divisor is contained in each part, and then the aggregate of these results is the complete quotient.

76. We shall now give some examples of Division arranged in a convenient form.

$$\begin{array}{l} a+b\,)\,a^2+2ab+b^2\,(\,a+b \\ \qquad\;\; a^2+ab \\ \qquad\;\; \overline{\qquad\quad ab+b^2} \\ \qquad\qquad\;\; ab+b^2 \\ \qquad\qquad\;\; \overline{\qquad\qquad} \end{array} \qquad\qquad \begin{array}{l} a+b\,)\,a^2-b^2\,(\,a-b \\ \qquad\;\; a^2+ab \\ \qquad\;\; \overline{\qquad -ab-b^2} \\ \qquad\qquad -ab-b^2 \\ \qquad\qquad \overline{\qquad\qquad} \end{array}$$

$$\begin{array}{l} a-b\,)\,a^2-b^2\,(\,a+b \\ \qquad\;\; a^2-ab \\ \qquad\;\; \overline{\qquad\quad ab-b^2} \\ \qquad\qquad\;\; ab-b^2 \\ \qquad\qquad\;\; \overline{\qquad\qquad} \end{array} \qquad\qquad \begin{array}{l} x^2+3x\,)\,x^3+2x^2-3x\,(\,x-1 \\ \qquad\qquad x^3+3x^2 \\ \qquad\qquad \overline{\qquad -x^2-3x} \\ \qquad\qquad\quad -x^2-3x \\ \qquad\qquad\quad \overline{\qquad\qquad} \end{array}$$

$$
\begin{array}{l}
a^2-2ab+3b^2\,)\,3a^4-10a^3b+22a^2b^2-22ab^3+15b^4\,(\,3a^2-4ab+5b^2 \\
\qquad\qquad\quad 3a^4-6a^3b+9a^2b^2 \\
\hline
\qquad\qquad\qquad -4a^3b+13a^2b^2-22ab^3 \\
\qquad\qquad\qquad -4a^3b+\ \ 8a^2b^2-12ab^3 \\
\hline
\qquad\qquad\qquad\qquad\qquad 5a^2b^2-10ab^3+15b^4 \\
\qquad\qquad\qquad\qquad\qquad 5a^2b^2-10ab^3+15b^4 \\
\hline
\end{array}
$$

Consider the last example. The dividend and divisor are both arranged according to descending powers of a. The first term in the dividend is $3a^4$ and the first term in the divisor is a^2; dividing the former by the latter we obtain $3a^2$ for the first term of the quotient. We then multiply the whole divisor by $3a^2$, and place the result so that each term comes below the term of the dividend which contains the same power of a; we subtract, and obtain $-4a^3b+13a^2b^2$; and we bring down the next term of the dividend, namely, $-22ab^3$. We divide the first term, $-4a^3b$, by the first term in the divisor, a^2; thus we obtain $-4ab$ for the next term in the quotient. We then multiply the whole divisor by $-4ab$ and place the result in order under those terms of the dividend with which we are now occupied; we subtract, and obtain $5a^2b^2-10ab^3$; and we bring down the next term of the dividend, namely, $15b^4$. We divide $5a^2b^2$ by a^2, and thus we obtain $5b^2$ for the next term in the quotient. We then multiply the whole divisor by $5b^2$, and place the terms as before; we subtract, and there is no remainder. As all the terms in the dividend have been brought down, the operation is completed; and the quotient is $3a^2-4ab+5b^2$.

It is of great importance to arrange both dividend and divisor according to the same order of some common letter; and to attend to this order in every part of the operation.

77. It may happen, as in Arithmetic, that the division *cannot be exactly performed.* Thus, for example, if we divide $a^2+2ab+2b^2$ by $a+b$, we shall obtain, as in the first example of the preceding Article, $a+b$ in the quotient, and there will *then be a remainder* b^2. This result is ex-

pressed in ways similar to those used in Arithmetic; thus we may say that

$$\frac{a^2+2ab+2b^2}{a+b}=a+b+\frac{b^2}{a+b};$$

that is, there is a quotient $a+b$, and a fractional part $\frac{b^2}{a+b}$.

In general, let A and B denote two expressions, and suppose that when A is divided by B the quotient is q, and the remainder R; then this result is expressed algebraically in the following ways,

$$A=qB+R, \text{ or } A-qB=R,$$

$$\text{or } \frac{A}{B}=q+\frac{R}{B}, \text{ or } \frac{A}{B}-q=\frac{R}{B}.$$

The student will observe that each letter here may represent an expression, simple or compound; it is often convenient for distinctness and brevity thus to represent an expression by a single letter.

We shall however consider algebraical fractions in subsequent Chapters, and at present shall confine ourselves to examples of Division in which the operation can be exactly performed.

78. We give some more examples:

Divide $x^7-5x^5+7x^3+2x^2-6x-2$ by $1+2x-3x^2+x^4$.

Arrange both dividend and divisor according to descending powers of x.

$$\begin{array}{r|l}
x^4-3x^2+2x+1 & x^7-5x^5+7x^3+2x^2-6x-2\,(x^3-2x-2 \\
 & \underline{x^7-3x^5+2x^4+x^3} \\
 & -2x^5-2x^4+6x^3+2x^2-6x \\
 & \underline{-2x^5+6x^3-4x^2-2x} \\
 & -2x^4+6x^2-4x-2 \\
 & \underline{-2x^4+6x^2-4x-2}
\end{array}$$

Divide $a^3+b^3+c^3-3abc$ by $a+b+c$.

Arrange the dividend according to descending powers of a.

$$
\begin{array}{l}
a+b+c\,)\,a^3 \qquad\qquad -3abc+b^3+c^3\,(\,a^2-ab-ac+b^2-bc+c^2 \\
\qquad\quad a^3+a^2b+a^2c \\
\hline
\qquad\quad -a^2b-a^2c \qquad -3abc \\
\qquad\quad -a^2b \qquad -ab^2-\ abc \\
\hline
\qquad\qquad\quad -a^2c+ab^2-2abc \\
\qquad\qquad\quad -a^2c \qquad -\ abc-ac^2 \\
\hline
\qquad\qquad\qquad\qquad ab^2-\ abc+ac^2+b^3 \\
\qquad\qquad\qquad\qquad ab^2 \qquad\qquad +b^3+b^2c \\
\hline
\qquad\qquad\qquad\qquad\quad -\ abc+ac^2 \quad -b^2c \\
\qquad\qquad\qquad\qquad\quad -\ abc \qquad\quad -b^2c-bc^2 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad ac^2 \qquad +bc^2+c^3 \\
\qquad\qquad\qquad\qquad\qquad\qquad ac^2 \qquad +bc^2+c^3 \\
\hline
\end{array}
$$

It will be seen that we arrange these terms according to descending powers of a; then when there are two terms, such as a^2b and a^2c, which involve the same power of a, we select a new letter, as b, and put the term which contains b before the term which does not; and again, of the terms ab^2 and abc, we put the former first as involving the higher power of b.

This example might also be worked, with the aid of brackets, thus:

$$
\begin{array}{l}
a+b+c\,)\,a^3 \qquad\qquad -3abc+b^3+c^3\,(\,a^2-a(b+c)+b^2-bc+c^2 \\
\qquad\quad a^3+a^2(b+c) \\
\hline
\qquad\quad -a^2(b+c)-3abc+b^3+c^3 \\
\qquad\quad -a^2(b+c)-a(b^2+2bc+c^2) \\
\hline
\qquad\qquad\qquad\qquad a(b^2-\ bc+c^2)+b^3+c^3 \\
\qquad\qquad\qquad\qquad a(b^2-\ bc+c^2)+b^3+c^3 \\
\hline
\end{array}
$$

Divide $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$ by $x-c$.

$$
\begin{array}{r|l}
x-c \,) & x^3-(a+b+c)x^2+(ab+ac+bc)x-abc\,(\,x^2-(a+b)x+ab \\
& x^3-cx^2 \\ \hline
& -(a+b)x^2+(ab+ac+bc)x-abc \\
& -(a+b)x^2+(a+b)cx \\ \hline
& abx \quad -abc \\
& abx \quad -abc \\ \hline
\end{array}
$$

Every example of Multiplication, in which the multiplier and the multiplicand are different expressions, will furnish two exercises in Division; because if the product be divided by either factor the quotient should be the other factor. Thus from the examples given in the section on Multiplication the student can derive exercises in Division, and test the accuracy of his work. And from any example of Division, in which the quotient and the divisor are different expressions, a second exercise may be obtained by making the quotient a divisor of the dividend, so that the new quotient ought to be the original divisor.

Examples. IX.

Divide

1. $15x^5$ by $3x^2$. 2. $24a^6$ by $-8a^3$. 3. $18x^3y^2$ by $6x^2y$.
4. $24a^4b^5c^6$ by $-3a^2b^3c^4$. 5. $20a^4b^4x^3y^3$ by $5b^2x^3y$.
6. $4x^3-8x^2+16x$ by $4x$. 7. $3a^4-12a^3+15a^2$ by $-3a^2$.
8. $x^3y-3x^2y^2+4xy^3$ by xy.
9. $-15a^3b^3-3a^2b^2+12ab$ by $-3ab$.
10. $60a^3b^3c^2-48a^2b^4c^2+36a^2b^2c^4-20abc^6$ by $4abc^2$.
11. $x^2-7x+12$ by $x-3$. 12. x^2+x-72 by $x+9$.
13. $2x^3-x^2+3x-9$ by $2x-3$.
14. $6x^3+14x^2-4x+24$ by $2x+6$.
15. $9x^3+3x^2+x-1$ by $3x-1$.
16. $7x^3-24x^2+58x-21$ by $7x-3$.

17. $x^6 - 1$ by $x - 1$. 18. $a^3 - 2ab^2 + b^3$ by $a - b$.

19. $x^4 - 81y^4$ by $x - 3y$.

20. $x^4 - 2x^3y + 2x^2y^2 - xy^3$ by $x - y$.

21. $x^5 - y^5$ by $x - y$. 22. $a^5 + 32b^5$ by $a + 2b$.

23. $2a^4 + 27ab^3 - 81b^4$ by $a + 3b$.

24. $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$ by $x^3 + y^3$.

25. $x^5 + 2x^4y + 3x^3y^2 - x^2y^3 - 2xy^4 - 3y^5$ by $x^3 - y^3$.

26. $x^4 - 5x^3 + 11x^2 - 12x + 6$ by $x^2 - 3x + 3$.

27. $x^4 + x^3 - 9x^2 - 16x - 4$ by $x^2 + 4x + 4$.

28. $x^4 - 13x^2 + 36$ by $x^2 + 5x + 6$.

29. $x^4 + 64$ by $x^2 + 4x + 8$.

30. $x^4 + 10x^3 + 35x^2 + 50x + 24$ by $x^2 + 5x + 4$.

31. $x^4 + x^3 - 24x^2 - 35x + 57$ by $x^2 + 2x - 3$.

32. $1 - x - 3x^2 - x^5$ by $1 + 2x + x^2$.

33. $x^6 - 2x^3 + 1$ by $x^2 - 2x + 1$.

34. $a^4 + 2a^2b^2 + 9b^4$ by $a^2 - 2ab + 3b^2$.

35. $a^6 - b^6$ by $a^3 - 2a^2b + 2ab^2 - b^3$.

36. $x^6 + 2x^5 - 4x^4 - 2x^3 + 12x^2 - 2x - 1$ by $x^2 + 2x - 1$.

37. $x^8 + 2x^6 + 3x^4 + 2x^2 + 1$ by $x^4 - 2x^3 + 3x^2 - 2x + 1$.

38. $x^{12} + x^6 - 2$ by $x^4 + x^2 + 1$.

39. $x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc$
by x^2 [illegible] $b)x + ab$.

40. $a^2x^4 + (2ac - b^2)x^2 + c^2$ by $ax^2 - bx +$ [illegible]

41. $x^4 - x^3y - xy^3 + y^4$ by $x^2 + xy + y^2$.

42. $x^3 - 3xy - y^3 - 1$ by $x - y - 1$.

43. $49x^2 + 21xy + 12yz - 16z^2$ by $7x + 3y - 4z$.

44. $a^2 + 2ab + b^2 - c^2$ by $a + b - c$.

45. $a^3 + 8b^3 + c^3 - 6abc$ by $a^2 + 4b^2 + c^2$ [illegible] $- 2ab - 2bc$.

46. $a^3 + 3a^2b + 3ab^2 + b^3 + c^3$ by $a + b + c$.

47. $a^2(b + c) + b^2(a - c) + c^2(a - b) + abc$ by $a + b + c$.

48. $x^3 - 2ax^2 + (a^2 + ab - b^2)x - a^2b + ab^2$ by $x - a + b$.

49. $(x + y)^2 - 2(x + y)z + z^2$ by $x + y - z$.

50. $(x + y)^3 + 3(x + y)^2z + 3(x + y)z^2 + z^3$
by $(x + y)^2 + 2(x + y)z + z^2$.

X. *General Results in Multiplication.*

79. There are some examples in Multiplication which occur so often in algebraical operations that they deserve especial notice.

The following three examples are of great importance.

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \quad +ab+b^2 \\ \hline a^2+2ab+b^2 \end{array} \qquad \begin{array}{l} a-b \\ a-b \\ \hline a^2-ab \\ \quad -ab+b^2 \\ \hline a^2-2ab+b^2 \end{array} \qquad \begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline a^2 \qquad -b^2 \end{array}$$

The first example gives the value of $(a+b)(a+b)$, that is, of $(a+b)^2$; thus we have

$$(a+b)^2 = a^2+2ab+b^2.$$

Thus *the square of the sum of two numbers is equal to the sum of the squares of the two numbers increased by twice their product.*

Again, the second example gives

$$(a-b)^2 = a^2-2ab+b^2.$$

Thus *the square of the difference of two numbers is equal to the sum of the squares of the two numbers diminished by twice their product.*

The last example gives

$$(a+b)(a-b) = a^2-b^2.$$

Thus *the product of the sum and difference of two numbers is equal to the difference of their squares.*

80. The results of the preceding Article furnish a simple example of one of the uses of Algebra; we may say that Algebra enables us to *prove general theorems respecting numbers*, and also to *express those theorems briefly.*

For example, the result $(a+b)(a-b)=a^2-b^2$ is proved to be true, and is expressed thus by symbols more compactly than by words.

A general result thus expressed by symbols is often called a *formula.*

81. We may here indicate the meaning of the sign $\pm$ which is made by combining the signs $+$ and $-$, and which is called the *double sign.*

Since $(a+b)^2=a^2+2ab+b^2$, and $(a-b)^2=a^2-2ab+b^2$, we may express these results in one formula thus:

$$(a \pm b)^2 = a^2 \pm 2ab + b^2,$$

where $\pm$ indicates that we may take either the sign $+$ or the sign $-$, *keeping throughout the upper sign or the lower sign.* $a \pm b$ is read thus, "*a plus or minus b.*"

82. We shall devote some Articles to explaining the use that can be made of the formulæ of Art. 79. We shall repeat these formulæ, and *number them for the sake of easy and distinct reference to them.*

$$(a+b)^2 = a^2+2ab+b^2 \quad (1)$$

$$(a-b)^2 = a^2-2ab+b^2 \quad (2)$$

$$(a+b)(a-b) = a^2-b^2 \quad (3)$$

83. The formulæ will sometimes be of use in Arithmetical calculations. For example; required the difference of the squares of 127 and 123. By the formula (3)

$$(127)^2-(123)^2=(127+123)(127-123)=250\times 4=1000.$$

Thus the required number is obtained more easily than it would be by squaring 127 and 123, and subtracting the second result from the first.

Again, by the formula (2)

$$(29)^2=(30-1)^2=900-60+1=841\,;$$

and thus the square of 29 is found more easily than by multiplying 29 by 29 directly.

Or suppose we have to multiply 53 by 47.

By the formula (3)

$$53\times 47=(50+3)(50-3)=(50)^2-3^2=2500-9=2491.$$

84. Suppose that we require the square of $3x+2y$. We can of course obtain it in the ordinary way, that is by multiplying $3x+2y$ by $3x+2y$. But we can also obtain it in another way, namely, by employing the formula (1). The formula is true whatever number a may be, and whatever number b may be; so we may put $3x$ for a, and $2y$ for b. Thus we obtain

$$(3x+2y)^2=(3x)^2+2(3x\,2y)+(2y)^2=9x^2+12xy+4y^2.$$

The beginner will probably think that in such a case he does not gain any thing by the use of the formula, for he will believe that he could have obtained the required result at least as easily and as safely by common work as by the use of the formula. This notion may be correct in this case, but it will be found that in more complex cases the formula will be of great service.

85. Suppose we require the square of $x+y+z$. Denote $x+y$ by a.

Then $x+y+z=a+z$; and by the use of (1) we have

$$(a+z)^2=a^2+2az+z^2=(x+y)^2+2(x+y)z+z^2$$
$$=x^2+2xy+y^2+2xz+2yz+z^2.$$

Thus $(x+y+z)^2=x^2+y^2+z^2+2xy+2yz+2xz$.

Suppose we require the square of $p-q+r-s$. Denote $p-q$ by a and $r-s$ by b; then $p-q+r-s=a+b$.

By the use of (1) we have

$$(a+b)^2=a^2+2ab+b^2=(p-q)^2+2(p-q)(r-s)+(r-s)^2.$$

Then by the use of (2) we express $(p-q)^2$ and $(r-s)^2$.

Thus $(p-q+r-s)^2$

$$=p^2-2pq+q^2+2(pr-ps-qr+qs)+r^2-2rs+s^2$$
$$=p^2+q^2+r^2+s^2+2pr+2qs-2pq-2ps-2qr-2rs.$$

Suppose we require the product of $p-q+r-s$ and $p-q-r+s$.

Let $p-q=a$ and $r-s=b$; then

$$p-q+r-s=a+b, \text{ and } p-q-r+s=a-b.$$

Then by the use of (3) we have

$$(a+b)(a-b)=a^2-b^2=(p-q)^2-(r-s)^2;$$

and by the use of (2) we have

$$(p-q+r-s)(p-q-r+s)=p^2-2pq+q^2-(r^2-2rs+s^2)$$
$$=p^2+q^2-r^2-s^2-2pq+2rs.$$

86. The method exhibited in the preceding Article is safe, and should therefore be adopted by the beginner; as he becomes more familiar with the subject he may dispense with some of the work. Thus in the last example, he will be able to omit that part relating to a and b, and simply put down the following process;

$$(p-q+r-s)(p-q-r+s)=\{p-q+(r-s)\}\{p-q-(r-s)\}$$
$$=(p-q)^2-(r-s)^2=p^2-2pq+q^2-(r^2-2rs+s^2)$$
$$=p^2-2pq+q^2-r^2+2rs-s^2;$$

or more briefly still,

$$(p-q+r-s)(p-q-r+s)=(p-q)^2-(r-s)^2$$
$$=p^2-2pq+q^2-r^2+2rs-s^2.$$

But at first the student will probably find it prudent to go through the work fully as in the preceding Article.

87. The following example will employ all the three formulæ.

Find the product of the four factors $a+b+c$, $a+b-c$, $a-b+c$, $b+c-a$.

Take the first two factors; by (3) and (1) we obtain

$$(a+b+c)(a+b-c)=(a+b)^2-c^2=a^2+2ab+b^2-c^2.$$

Take the last two factors; by (3) and (2) we obtain

$$(a-b+c)(b+c-a)=\{c+(a-b)\}\{c-(a-b)\}$$
$$=c^2-(a-b)^2=c^2-a^2+2ab-b^2.$$

We have now to multiply together $a^2+2ab+b^2-c^2$ and $c^2-a^2+2ab-b^2$. We obtain

$$(a^2+2ab+b^2-c^2)(c^2-a^2+2ab-b^2)$$
$$=\{2ab+(a^2+b^2-c^2)\}\{2ab-(a^2+b^2-c^2)\}$$
$$=(2ab)^2-(a^2+b^2-c^2)^2$$
$$=4a^2b^2-\{(a^2+b^2)^2-2(a^2+b^2)c^2+c^4\}$$
$$=4a^2b^2-(a^2+b^2)^2+2(a^2+b^2)c^2-c^4$$
$$=4a^2b^2-a^4-2a^2b^2-b^4+2a^2c^2+2b^2c^2-c^4$$
$$=2a^2b^2+2b^2c^2+2a^2c^2-a^4-b^4-c^4.$$

88. There are other results in Multiplication which are of less importance than the three formulæ given in Art. 82, but which are deserving of attention. We place them here in order that the student may be able to refer to them when they are wanted; they can be easily verified by actual multiplication.

$$(a+b)(a^2-ab+b^2)=a^3+b^3,$$
$$(a-b)(a^2+ab+b^2)=a^3-b^3,$$
$$(a+b)^3=(a+b)(a^2+2ab+b^2)=a^3+3a^2b+3ab^2+b^3,$$
$$(a-b)^3=(a-b)(a^2-2ab+b^2)=a^3-3a^2b+3ab^2-b^3,$$
$$(a+b+c)^3=a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3,$$
$$=a^3+3a^2(b+c)+3a(b^2+2bc+c^2)+b^3+3b^2c+3bc^2+c^3$$
$$=a^3+b^3+c^3+3a^2(b+c)+3b^2(a+c)+3c^2(a+b)+6abc.$$

89. Useful exercises in Multiplication are formed by requiring the student to shew that two expressions agree in giving the same result. For example, shew that

$$(a-b)(b-c)(c-a)=a^2(c-b)+b^2(a-c)+c^2(b-a).$$

If we multiply $a-b$ by $b-c$ we obtain

$$ab-b^2-ac+bc;$$

then by multiplying this result by $c-a$ we obtain

$$cab-cb^2-ac^2+bc^2-a^2b+ab^2+a^2c-abc,$$
$$\text{that is } a^2(c-b)+b^2(a-c)+c^2(b-a).$$

Again; shew that $(a-b)^2+(b-c)^2+(c-a)^2$

$$=2(c-b)(c-a)+2(b-a)(b-c)+2(a-b)(a-c).$$

By using formula (2) of Art. 82 we obtain

$$(a-b)^2+(b-c)^2+(c-a)^2$$
$$=a^2-2ab+b^2+b^2-2bc+c^2+c^2-2ac+a^2$$
$$=2(a^2+b^2+c^2-ab-ac-bc).$$

And
$$(c-b)(c-a)=c^2-ca-cb+ab,$$
$$(b-a)(b-c)=b^2-ba-bc+ac,$$
$$(a-b)(a-c)=a^2-ab-ac+bc;$$

therefore $(c-b)(c-a)+(b-a)(b-c)+(a-b)(a-c)$
$$=a^2+b^2+c^2-ab-ac-bc;$$

therefore $(a-b)^2+(b-c)^2+(c-a)^2$
$$=2(c-b)(c-a)+2(b-a)(b-c)+2(a-b)(a-c).$$

EXAMPLES. X.

Apply the formulæ of Art. 82 to the following sixteen examples in multiplication:

1. $(15x+14y)^2$.
2. $(7x^2-5y^2)^2$.
3. $(x^2+2x-2)^2$.
4. $(x^2-5x+7)^2$.
5. $(2x^2-3x-4)^2$.
6. $(x+2y+3z)^2$.
7. $(x^2+xy+y^2)(x^2+xy-y^2)$.
8. $(x^2+xy+y^2)(x^2-xy+y^2)$.
9. $(x^2+xy+y^2)(x^2-xy-y^2)$.
10. $(x^2+xy-y^2)(x^2-xy+y^2)$.
11. $(x^3+2x^2+3x+1)(x^3-2x^2+3x-1)$.
12. $(x-3)^2(x^2+6x+9)$.
13. $(a+b)^2(a^2-2ab-b^2)$.
14. $(2x+3y)^2(4x^2+12xy-9y^2)$.
15. $(ax+by)(ax-by)(a^2x^2+b^2y^2)$.
16. $(ax+by)^2(ax-by)^2$.

Shew that the following results are true:

17. $(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2$.

18. $(a+b+c)^2+a^2+b^2+c^2=(a+b)^2+(b+c)^2+(c+a)^2$.

19. $(a-b)(b-c)(c-a)=bc(c-b)+ca(a-c)+ab(b-a)$.

20. $(a-b)^3+b^3-a^3=3ab(b-a)$.

21. $(a+b+c)^2-a(b+c-a)-b(a+c-b)-c(a+b-c)$
$=2(a^2+b^2+c^2)$.

22. $(a^2+ab+b^2)^2-(a^2-ab+b^2)^2=4ab(a^2+b^2)$.

23. $(a+b+c)^3-a^3-b^3-c^3=3(a+b)(b+c)(c+a)$.

24. $(a+b+c)(ab+bc+ca)=(a+b)(b+c)(c+a)+abc$.

25. $(a+b)(b+c-a)(c+a-b)$
$=a(b^2+c^2-a^2)+b(c^2+a^2-b^2)$.

26. $(a+b+c)^3-(b+c-a)^3-(a-b+c)^3-(a+b-c)^3$
$=24abc$.

27. $(a+b+c)^2+(a+b-c)^2+(a-b+c)^2+(b+c-a)^2$
$=4(a^2+b^2+c^2)$.

28. $(a+b)^2+2(a^2-b^2)+(a-b)^2=(2a)^2$.

29. $(a-b)^3+(b-c)^3+(c-a)^3=3(a-b)(b-c)(c-a)$.

30. $(a-b)^3+(a+b)^3+3(a-b)^2(a+b)+3(a+b)^2(a-b)$
$=(2a)^3$.

31. $(a+b)^2(b+c-a)(c+a-b)+(a-b)^2(a+b+c)(a+b-c)$
$=4abc^2$.

32. $a(b+c)(b^2+c^2-a^2)+b(c+a)(c^2+a^2-b^2)$
$+c(a+b)(a^2+b^2-c^2)=2abc(a+b+c)$.

33. $(a-b)(x-a)(x-b)+(b-c)(x-b)(x-c)$
$+(c-a)(x-c)(x-a)=(a-b)(b-c)(a-c)$.

34. $(a+b)^2+(a+c)^2+(a+d)^2+(b+c)^2+(b+d)^2+(c+d)^2$
$=(a+b+c+d)^2+2(a^2+b^2+c^2+d^2)$.

35. $\{(ax+by)^2+(ay-bx)^2\}\{(ax+by)^2-(ay+bx)^2\}$
$=(a^4-b^4)(x^4-y^4)$.

$(cy-bz)^2+(az-cx)^2+(bx-ay)^2+(ax+by+cz)^2$
$=(a^2+b^2+c^2)(x^2+y^2+z^2)$.

XI. *Factors.*

90. In the preceding Chapter we have noticed some general results in Multiplication; these results may also be regarded in connexion with Division, because every example in Multiplication furnishes an example or examples in Division. We shall now apply some of these results to find what expressions will divide a given expression, or in other words to *resolve expressions into their factors.*

91. For example, by the use of formula (3) of Art. 82 we have

$$a^4-b^4=(a^2+b^2)(a^2-b^2)=(a^2+b^2)(a+b)(a-b);$$

$$a^8-b^8=(a^4+b^4)(a^4-b^4)=(a^4+b^4)(a^2+b^2)(a+b)(a-b).$$

Hence we see that a^8-b^8 is the product of the four factors a^4+b^4, a^2+b^2, $a+b$, and $a-b$. Thus a^8-b^8 is divisible by any of these factors, or by the product of any two of them, or by the product of any three of them.

Again,

$$(a^2+ab+b^2)(a^2-ab+b^2)=(a^2+b^2+ab)(a^2+b^2-ab)$$

$$=(a^2+b^2)^2-(ab)^2=a^4+2a^2b^2+b^4-a^2b^2=a^4+a^2b^2+b^4.$$

Thus $a^4+a^2b^2+b^4$ is the product of the two factors a^2+ab+b^2 and a^2-ab+b^2, and is therefore divisible by either of them.

Besides the results which we have already given, we shall now place a few more before the student.

92. The following examples in division may be easily verified.

$$\frac{x-y}{x-y}=1,$$

$$\frac{x^2-y^2}{x-y}=x+y,$$

$$\frac{x^3-y^3}{x-y}=x^2+xy+y^2,$$

$$\frac{x^4-y^4}{x-y}=x^3+x^2y+xy^2+y^3,$$

and so on.

Also

$$\frac{x^2-y^2}{x+y}=x-y,$$

$$\frac{x^4-y^4}{x+y}=x^3-x^2y+xy^2-y^3,$$

$$\frac{x^6-y^6}{x+y}=x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5,$$

and so on.

Also

$$\frac{x+y}{x+y}=1,$$

$$\frac{x^3+y^3}{x+y}=x^2-xy+y^2,$$

$$\frac{x^5+y^5}{x+y}=x^4-x^3y+x^2y^2-xy^3+y^4,$$

and so on.

The student can carry on these operations as far as he pleases, and he will thus gain confidence in the truth of the statements which we shall now make, and which are strictly demonstrated in the higher parts of larger works on Algebra. The following are the statements:

x^n-y^n is divisible by $x-y$ if n be *any* whole number;

x^n-y^n is divisible by $x+y$ if n be any *even* whole number;

x^n+y^n is divisible by $x+y$ if n be any *odd* whole number.

We might also put into words a statement of the forms of the quotient in the three cases; but the student will most readily learn these forms by looking at the above examples and, if necessary, carrying the operations still farther.

We may add that x^n+y^n is never divisible by $x+y$ or $x-y$, when n is an *even* whole number.

93. The student will be assisted in remembering the results of the preceding Article by noticing the simplest

case in each of the four results, and referring other cases to it. For example, suppose we wish to consider whether x^7-y^7 is divisible by $x-y$ or by $x+y$; the index 7 is an *odd* whole number, and the simplest case of this kind is $x-y$, which is divisible by $x-y$, but not by $x+y$; so we infer that x^7-y^7 is divisible by $x-y$ and not by $x+y$. Again, take x^8-y^8; the index 8 is an *even* whole number, and the simplest case of this kind is x^2-y^2, which is divisible both by $x-y$ and $x+y$; so we infer that x^8-y^8 is divisible both by $x-y$ and $x+y$.

94. The following are additional examples of resolving expressions into factors.

$$x^6-y^6=(x^3+y^3)(x^3-y^3)$$
$$=(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2);$$

$$8b^3-27c^3=(2b)^3-(3c)^3=(2b-3c)\{(2b)^2+2b\times 3c+(3c)^2\}$$
$$=(2b-3c)(4b^2+6bc+9c^2);$$

$$4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2=$$
$$\{2(ab+cd)+(a^2+b^2-c^2-d^2)\}\{2(ab+cd)-(a^2+b^2-c^2-d^2)\}$$
$$=\{2ab+2cd+a^2+b^2-c^2-d^2\}\{2ab+2cd-a^2-b^2+c^2+d^2\}$$
$$=\{(a+b)^2-(c-d)^2\}\{(c+d)^2-(a-b)^2\}$$
$$=(a+b+c-d)(a+b-c+d)(a-b+c+d)(b+c+d-a).$$

95. Suppose that $(x^2-5xy+6y^2)(x-4y)$ is to be divided by $x^2-7xy+12y^2$. We might multiply $x^2-5xy+6y^2$ by $x-4y$, and then divide the result by $x^2-7xy+12y^2$. But the form of the question suggests to us to try if $x-4y$ is not a factor of $x^2-7xy+12y^2$; and we shall find that $x^2-7xy+12y^2=(x-3y)(x-4y)$. Then

$$\frac{(x^2-5xy+6y^2)(x-4y)}{(x-3y)(x-4y)}=\frac{x^2-5xy+6y^2}{x-3y};$$

and by division we find that

$$\frac{x^2-5xy+6y^2}{x-3y}=x-2y.$$

96. The student with a little practice will be able to resolve certain trinomials into two binomial factors.

For we have generally

$$(x+a)(x+b)=x^2+(a+b)x+ab;$$

suppose then we wish to know if it be possible to resolve $x^2+7x+12$ into two binomial factors; we must find, if possible, two numbers such that their sum is 7 and their product is 12; and we see that 3 and 4 are such numbers. Thus

$$x^2+7x+12=(x+3)(x+4).$$

Similarly, by the aid of the *formula*

$$(x-a)(x-b)=x^2-(a+b)x+ab,$$

we can resolve $x^2-7x+12$ into the factors $(x-3)(x-4)$.

And, by the aid of the formula

$$(x+a)(x-b)=x^2+(a-b)x-ab,$$

we can resolve x^2+x-12 into the factors $(x+4)(x-3)$.

We shall now give for exercise some miscellaneous examples in the preceding Chapters.

Examples. XI.

Add together the following expressions:

1. $a(a+b-c)$, $b(b+c-a)$, $c(a+c-b)$.
2. $a(a-b+c)$, $b(b-c+a)$, $c(c-a+b)$.
3. $a(a-b+c+d)$, $b(a+b-c+d)$, $c(a+b+c-d)$, $d(-a+b+c+d)$.
4. $3a-(4b-7c)$, $3b-(4c-7a)$, $3c-(4a-7b)$.
5. $9a-(5b+2c)$, $9b-(5c+2a)$, $9c-(5a+2b)$.
6. $(a+b)x+(a+c)y$, $(b-c)x+(b-c)y$, $(c-a)x+(b-a)y$.

7. $(z-x)(a+b)+(z-y)(a-b)$, $(x+y)a+(x+z)b$, $(y-z)a+(x-y)b$.

8. $(a-b)x+(b-c)y+(c-a)z$, $a(y+z)+b(z+x)+c(x+y)$, $ax+by+cz$.

9. $2(a+b-c)x+(a+b)y+2az$, $2(a+c-b)x+(a+c)y+2bz$, $2(b+c-a)x+(b+c)y+2cz$.

10. $a^2-(a-b+c)(a+b-c)$, $b^2-(b-a+c)(b+a-c)$, $c^2-(c-a+b)(c+a-b)$.

Simplify the following expressions:

11. $a-2(b+3a)-3\{b+2(a-b)\}$.

12. $(a+b)(b+c)-(c+d)(d+a)-(a+c)(b-d)$.

13. $4a-[2a-\{2b(x+y)-2b(x-y)\}]$.

14. $(x+b)(x+c)-(a+b+c)(x+b)+a^2+ab+b^2+3ax$.

15. $a-[5b-\{a-3(c-b)+2c-(a-2b-c)\}]$.

16. $5a-7(b-c)-[6a-(3b+2c)+4c-\{2a-(b+c-a)\}]$.

17. $(x+3)^3-3(x+2)^3+3(x+1)^3-x^3$.

18. $(x+y)^3+(x+y)^2y+(x+y)y^2-\{3x^2y+5y^2x+2y^3\}$.

19. $(1+x)^3+(1+x)^2y+(1+x)y^2+y^3$
$-\{3x(x+1)+y(y+1)+2xy+1\}$.

20. $a(b+c)^2+b(a+c)^2+c(a+b)^2+(a-b)(a+c)(b-c)$
$-(a+b)(a-c)(b-c)-(a-b)(a-c)(b+c)$.

21. $\dfrac{(a+b)(a+c)-(b+d)(d+c)}{a-d}$.

22. $\dfrac{a^2-3ab+2b^2}{a-2b}-\dfrac{a^2-7ab+12b^2}{a-3b}$.

23. $\dfrac{3a^3-7a^2b-5ab^2+5b^3}{a+b}+\dfrac{6a^3-26a^2b+40ab^2-20b^3}{a-b}$.

24. $\dfrac{18(bc^2+ca^2+ab^2)-12(b^2c+c^2a+a^2b)-19abc}{2a-3b}$.

Divide

25. $x^6+y^6-2x^3y^3$ by $(x-y)^2$.

26. $x^6+y^6+2x^3y^3$ by $(x+y)^2$.

27. $(a^3-3a^2b+5ab^2-3b^3)(a-2b)$ by $a^2-3ab+2b^2$.

28. $(x^3-9x^2y+23xy^2-15y^3)(x-7y)$ by $x^2-8xy+7y^2$.

29. $a^8+a^4b^4+b^8$ by $(a^2-ab+b^2)(a^2+ab+b^2)$.

30. $a^8-b^8+a^2b^2(a^4-b^4)$ by $(a^2-ab+b^2)(a^2+ab+b^2)$.

31. $4a^2b^2+2(3a^4-2b^4)-ab(5a^2-11b^2)$ by $(3a-b)(a+b)$.

32. $(x^2-3x+2)(x-3)$ by x^2-5x+6.

33. $(x^3-3x+2)(x+4)$ by x^2+x-2.

34. $(a^2+ax+x^2)(a^3+x^3)$ by $a^4+a^2x^2+x^4$.

35. $(a^4+a^2b^2+b^4)(a+b)$ by a^2+ab+b^2.

36. $b(x^3+a^3)+ax(x^2-a^2)+a^3(x+a)$ by $(a+b)(x+a)$.

Resolve the following expressions into factors:

37. $x^2+9x+20$.

38. $x^2+11x+30$.

39. $x^2-15x+50$.

40. $x^2-20x+100$.

41. $x^2+x-132$.

42. $x^2-7x-44$.

43. x^4-81.

44. x^3+125.

45. x^8-256.

46. x^6-64.

47. $a^2+9ab+20b^2$.

48. $x^2-13xy+42y^2$.

49. $(a+b)^2-11c(a+b)+30c^2$.

50. $2(x+y)^2-7(x+y)(a+b)+3(a+b)^2$.

Shew that the following results are true:

51. $(a+2b)a^3-(b+2a)b^3=(a-b)(a+b)^3$.

52. $a(a-2b)^3-b(b-2a)^3=(a-b)(a+b)^3$.

XII. *Greatest Common Measure.*

97. In Arithmetic a whole number which divides another whole number exactly is said to be a *measure* of it, or to *measure* it; a whole number which divides two or more whole numbers exactly is said to be a *common measure* of them.

In Algebra an expression which divides another expression exactly is said to be a *measure* of it, or to *measure* it; an expression which divides two or more expressions exactly is said to be a *common measure* of them.

98. In Arithmetic the *greatest common measure* of two or more whole numbers is the greatest whole number which will measure them all. The term greatest common measure is also used in Algebra, but here it is not very appropriate, because the terms *greater* and *less* are seldom applicable to those algebraical expressions in which definite numerical values have not been assigned to the various letters which occur. It would be better to speak of the *highest common measure*, or of the *highest common divisor;* but in conformity with established usage we shall retain the term *greatest common measure.*

The letters G.C.M. will often be used for shortness instead of this term.

We have now to explain in what sense the term is used in Algebra.

99. It is usual to say, that by the greatest common measure of two or more simple expressions is meant the *greatest expression which will measure them all;* but this definition will not be fully understood until we have given and exemplified the rule for finding the greatest common measure of simple expressions.

The following is the Rule for finding the G.C.M. of simple expressions. *Find by Arithmetic the* G.C.M. *of the numerical coefficients; after this number put every letter which is common to all the expressions, and give to each letter respectively the least index which it has in the expressions.*

100. For example; required the G.C.M. of $16a^4b^2c$ and $20a^3b^3d$. Here the numerical coefficients are 16 and 20, and their G.C.M. is 4. The letters common to both the expressions are a and b; the least index of a is 3, and the least index of b is 2. Thus we obtain $4a^3b^2$ as the required G.C.M.

Again; required the G.C.M. of $8a^2b^3c^2x^5yz^3$, $12a^4bcx^2y^3$, and $16a^3c^3x^2y^4$. Here the numerical coefficients are 8, 12, and 16; and their G.C.M. is 4. The letters common to all the expressions are a, c, x, and y; and their least indices are respectively 2, 1, 2, and 1. Thus we obtain $4a^2cx^2y$ as the required G.C.M.

101. The following statement gives the best practical notion of what is meant by the term greatest common measure, in Algebra, as it shews the sense of the word *greatest* here. *When two or more expressions are divided by their greatest common measure, the quotients have no common measure.*

Take the first example of Art. 100, and divide the expressions by their G.C.M.; the quotients are $4ac$ and $5bd$, and these quotients have no common measure.

Again, take the second example of Art. 100, and divide the expressions by their G.C.M.; the quotients are $2b^3cx^3z^3$, $3a^2by^2$, and $4ac^2y^3$, and these quotients have no common measure.

102. The notion which is supplied by the preceding Article, with the aid of the Chapter on Factors, will enable the student to determine in many cases the G.C.M. of *compound expressions.* For example; required the G.C.M. of $4a^2(a+b)^2$ and $6ab(a^2-b^2)$. Here $2a$ is the G.C.M. of the factors $4a^2$ and $6ab$; and $a+b$ is a factor of $(a+b)^2$ and of a^2-b^2, and is the only common factor. The product $2a(a+b)$ is then the G.C.M. of the given expressions.

But this method cannot be applied to complex examples, because the general theory of the resolution of expressions into factors is beyond the present stage of the student's knowledge; it is therefore necessary to adopt

another method, and we shall now give the usual definition and rule.

103. The following may be given as the definition of the greatest common measure of compound expressions. *Let two or more compound expressions contain powers of some common letter; then the factor of highest dimensions in that letter which divides all the expressions is called their greatest common measure.*

104. The following is the Rule for finding the greatest common measure of two compound expressions.

Let A and B denote the two expressions; let them be arranged according to descending powers of some common letter, and suppose the index of the highest power of that letter in A not less than the index of the highest power of that letter in B. Divide A by B; then make the remainder a divisor and B the dividend. Again make the new remainder a divisor and the preceding divisor the dividend. Proceed in this way until there is no remainder; then the last divisor is the greatest common measure required.

105. For example; required the G.C.M. of x^2-4x+3 and $4x^3-9x^2-15x+18$.

$$
\begin{array}{r}
x^2-4x+3)\ 4x^3-9x^2-15x+18\ (4x+7 \\
4x^3-16x^2+12x\phantom{{}+18\ (4x+7} \\
\hline
7x^2-27x+18 \\
7x^2-28x+21 \\
\hline
x-3
\end{array}
$$

$$
\begin{array}{r}
x-3)\ x^2-4x+3\ (x-1 \\
x^2-3x\phantom{{}+3\ (x-1} \\
\hline
-x+3 \\
-x+3 \\
\hline
\end{array}
$$

Thus $x-3$ is the G.C.M. required.

106. The rule which is given in Art. 104 depends on the following two principles.

(1) If P measure A, it will measure mA. For let a denote the quotient when A is divided by P; then $A=aP$; therefore $mA=maP$; therefore P measures mA.

(2) If P measure A and B, it will measure $mA \pm nB$. For, since P measures A and B, we may suppose $A=aP$, and $B=bP$; therefore $mA \pm nB=(ma \pm nb)P$; therefore P measures $mA \pm nB$.

107. We can now demonstrate the rule which is given in Art. 104.

Let A and B denote the two expressions. Divide A by B; let p denote the quotient, and C the remainder. Divide B by C; let q denote the quotient, and D the remainder. Divide C by D, and suppose that there is no remainder, and let r denote the quotient.

$$\begin{array}{l} B)\ A\ (p \\ \quad pB \\ \hline \quad C)\ B\ (q \\ \qquad qC \\ \hline \qquad D)\ C\ (r \\ \qquad\quad rD \\ \hline \end{array}$$

Thus we have the following results:

$$A=pB+C, \qquad B=qC+D, \qquad C=rD.$$

We shall first shew that D is *a* common measure of A and B. Because $C=rD$, therefore D measures C; therefore, by Art. 106, D measures qC, and also $qC+D$; that is, D measures B. Again, since D measures B and C, it measures $pB+C$; that is, D measures A. Thus D measures A and B.

We have thus shewn that D is *a* common measure of A and B; we shall now shew that it is their *greatest* common measure.

By Art. 106 every common measure of A and B measures $A-pB$, that is C; thus every common measure of A and B is a common measure of B and C. Similarly, every common measure of B and C is a common measure

of C and D. Therefore every common measure of A and B is a measure of D. But no expression of higher dimensions than D can divide D. Therefore D is the *greatest* common measure of A and B.

108. It is obvious that, *every measure of a common measure of two or more expressions is a common measure of those expressions.*

109. It is shewn in Art. 107 that every common measure of A and B measures D; that is, *every common measure of two expressions measures their greatest common measure.*

110. We shall now state and exemplify a rule which is adopted in order to avoid fractions in the quotient; by the use of the rule the work is simplified. We refer to the Chapter on the Greatest Common Measure in the larger Algebra, for the demonstration of the rule.

Before placing a fresh term in any quotient, *we may divide the divisor, or the dividend, by any expression which has no factor which is common to the expressions whose greatest common measure is required;* or, *we may multiply the dividend at such a stage by any expression which has no factor that occurs in the divisor.*

111. For example; required the G.C.M. of $2x^2-7x+5$ and $3x^2-7x+4$. Here we take $2x^2-7x+5$ as divisor; but if we divide $3x^2$ by $2x^2$ the quotient is a fraction; to avoid this we multiply the dividend by 2, and then divide.

$$
\begin{array}{r|l}
2x^2-7x+5) & 6x^2-14x+\ \ 8\ (3 \\
 & 6x^2-21x+15 \\
\hline
 & \qquad 7x-\ \ 7
\end{array}
$$

If we now make $7x-7$ a divisor and $2x^2-7x+5$ the dividend, the first term of the quotient will be fractional; but the factor 7 occurs in every term of the proposed divisor, and we remove this, and then divide.

$$
\begin{array}{l}
x-1)\ 2x^2-7x+5\ (2x-5 \\
\qquad\ \ 2x^2-2x \\
\hline
\qquad\qquad -5x+5 \\
\qquad\qquad -5x+5 \\
\hline
\end{array}
$$

Thus we obtain $x-1$ as the G.C.M. required.

Here it will be seen that we used the second part of the rule of Art. 110, at the beginning of the process, and the first part of the rule later. The first part of the rule should be used if possible; and if not, the second part. We have used the word *expression* in stating the rule, but in the examples which the student will have to solve, the factors introduced or removed will be almost always *numerical factors*, as they are in the preceding example.

We will now give another example; required the G.C.M. of $2x^4-7x^3-4x^2+x-4$ and $3x^4-11x^3-2x^2-4x-16$.

Multiply the latter expression by 2 and then take it for dividend.

$$
\begin{array}{l}
2x^4-7x^3-4x^2+x-4)\ 6x^4-22x^3-4x^2-8x-32\ (3 \\
\qquad\qquad\qquad\qquad\quad 6x^4-21x^3-12x^2+3x-12 \\
\hline
\qquad\qquad\qquad\qquad\qquad\ \ -x^3+8x^2-11x-20
\end{array}
$$

We may multiply every term of this remainder by -1 before using it as a new divisor; that is, we may change the sign of every term.

$$
\begin{array}{l}
x^3-8x^2+11x+20)\ 2x^4-7x^3-4x^2+x-4\ (2x+9 \\
\qquad\qquad\qquad\quad 2x^4-16x^3+22x^2+40x \\
\hline
\qquad\qquad\qquad\qquad\quad 9x^3-26x^2-39x-4 \\
\qquad\qquad\qquad\qquad\quad 9x^3-72x^2+99x+180 \\
\hline
\qquad\qquad\qquad\qquad\qquad\quad 46x^2-138x-184
\end{array}
$$

Here 46 is a factor of every term of the remainder; we remove it before using the remainder as a new divisor.

$$
\begin{array}{r}
x^2-3x-4\,)\ x^3-8x^2+11x+20\ (x-5 \\
x^3-3x^2-\ \ 4x \qquad\quad \\
\hline
-5x^2+15x+20 \\
-5x^2+15x+20 \\
\hline
\end{array}
$$

Thus x^2-3x-4 is the G.C.M. required.

112. Suppose the original expressions to contain a common factor F, which is obvious on inspection; let $A=aF$ and $B=bF$. Then, by Art. 109, F will be a factor of the G.C.M. Find the G.C.M. of a and b, and multiply it by F; the product will be the G.C.M. of A and B.

113. We now proceed to the G.C.M. of more than two compound expressions. Suppose we require the G.C.M. of *three* expressions A, B, C. Find the G.C.M. of any two of them, say of A and B; let D denote this G.C.M.; then the G.C.M. of D and C will be the required G.C.M. of A, B, and C.

For, by Art. 108, every common measure of D and C is a common measure of A, B, and C; and by Art. 109 every common measure of A, B, and C is a common measure of D and C. Therefore the G.C.M. of D and C is the G.C.M. of A, B, and C.

114. In a similar manner we may find the G.C.M. of *four* expressions. Or we may find the G.C.M. of two of the given expressions, and also the G.C.M. of the other two; then the G.C.M. of the two results thus obtained will be the G.C.M. of the four given expressions.

EXAMPLES. XII.

Find the greatest common measure in the following examples:

1. $15x^4$, $18x^2$.
2. $16a^2b^3$, $20a^3b^2$.
3. $36x^4y^5z^6$, $48x^6y^5z^4$.
4. $35a^2b^3x^3y^4$, $49a^2b^4x^4y^3$
5. $4(x+1)^2$, $6(x^2-1)$.
6. $6(x+1)^3$, $9(x^2-1)$.

7. $12(a^2+b^2)^2$, $8(a^4-b^4)$. 8. x^6-y^6, x^4-y^4.

9. $x^2+8x+15$, $x^2+9x+20$.

10. $x^2-9x+14$, $x^2-11x+28$.

11. $x^2+2x-120$, $x^2-2x-80$.

12. $x^2-15x+36$, $x^2-9x-36$.

13. $x^3+6x^2+13x+12$, $x^3+7x^2+16x+16$.

14. $x^3-9x^2+23x-12$, $x^3-10x^2+28x-15$.

15. $x^3-29x+42$, $x^3+x^2-35x+49$.

16. $x^3-41x-30$, $x^3-11x^2+25x+25$.

17. $x^3+7x^2+17x+15$, $x^3+8x^2+19x+12$.

18. $x^3-10x^2+26x-8$, $x^3-9x^2+23x-12$.

19. $4(x^2-x+1)$, $3(x^4+x^2+1)$.

20. $5(x^2-x+1)$, $4(x^6-1)$.

21. $6x^2+x-2$, $9x^3+48x^2+52x+16$.

22. x^3-4x^2+2x+3, $2x^4-9x^3+12x^2-7$.

23. x^4+x^2-6, x^4-3x^2+2.

24. x^3-2x^2+3x-6, $x^4-x^3-x^2-2x$.

25. x^4-1, $3x^5+2x^4+4x^3+2x^2+x$.

26. $x^4-9x^2-30x-25$, $x^5+x^4-7x^2+5x$.

27. $35x^3+47x^2+13x+1$, $42x^4+41x^3-9x^2-9x-1$.

28. $x^6-3x^5+6x^4-7x^3+6x^2-3x+1$,
$x^6-x^5+2x^4-x^3+2x^2-x+1$.

29. $2x^4-6x^3+3x^2-3x+1$, $x^7-3x^6+x^5-4x^2+12x-4$.

30. x^8-1, $x^{10}+x^9+x^8+2x^7+2x^4+2x^3+x^2+x+1$.

31. $x^2-3x-70$, $x^3-39x+70$, $x^3-48x+7$.

32. $x^2-xy-12y^2$, $x^2+5xy+6y^2$.

33. $2x^2+3ax+a^2$, $3x^2+2ax-a^2$.

34. $x^3-3a^2x-2a^3$, $x^3-ax^2-4a^3$.

35. $3x^3-3x^2y+xy^2-y^3$, $4x^2y-5xy^2+y^3$.

XIII. *Least Common Multiple.*

115. In Arithmetic a whole number which is measured by another whole number is said to be a multiple of it; a whole number which is measured by two or more whole numbers is said to be a *common multiple* of them.

116. In Arithmetic the *least common multiple* of two or more whole numbers is the least whole number which is measured by them all. The term least common multiple is also used in Algebra, but here it is not very appropriate; see Art. 98. The letters L.C.M. will often be used for shortness instead of this term.

We have now to explain in what sense the term is used in Algebra.

117. It is usual to say, that by the least common multiple of two or more simple expressions, is meant the *least expression which is measured by them all;* but this definition will not be fully understood until we have given and exemplified the rule for finding the least common multiple of simple expressions.

The following is the Rule for finding the L.C.M. of simple *expressions. Find by Arithmetic the* L.C.M. *of the numerical coefficients; after this number put every letter which occurs in the expressions, and give to each letter respectively the greatest index which it has in the expressions.*

118. For example; required the L.C.M. of $16a^4bc$ and $20a^3b^3d$. Here the numerical coefficients are 16 and 20, and their L.C.M. is 80. The letters which occur in the expressions are a, b, c, and d; and their greatest indices are respectively 4, 3, 1, and 1. Thus we obtain $80a^4b^3cd$ as the required L.C.M.

Again; required the L.C.M. of $8a^2b^3c^2x^5yz^3$, $12a^4bcx^2y^3$, and $16a^3c^3x^2y^4$. Here the L.C.M. of the numerical coefficients is 48. The letters which occur in the expressions are a, b, c, x, y, and z; and their greatest indices are respectively 4, 3, 3, 5, 4, and 3. Thus we obtain $48a^4b^3c^3x^5y^4z^3$ as the required L.C.M.

119. The following statement gives the best practical notion of what is meant by the term least common multiple in Algebra, as it shews the sense of the word *least* here. *When the least common multiple of two or more expressions is divided by those expressions the quotients have no common measure.*

Take the first example of Art. 118, and divide the L.C.M. by the expressions; the quotients are $5b^2d$ and $4ac$, and these quotients have no common measure.

Again; take the second example of Art. 118, and divide the L.C.M. by the expressions; the quotients are $6a^2cy^3$, $4b^2c^2x^3yz^3$, and $3ab^3x^3z^3$, and these quotients have no common measure.

120. The notion which is supplied by the preceding Article, with the aid of the Chapter on Factors, will enable the student to determine in many cases the L.C.M. of *compound expressions.* For example, required the L.C.M. of $4a^2(a+b)^2$ and $6ab(a^2-b^2)$. The L.C.M. of $4a^2$ and $6ab$ is $12a^2b$. Also $(a+b)^2$ and a^2-b^2 have the common factor $a+b$, so that $(a+b)(a+b)(a-b)$ is a multiple of $(a+b)^2$ and of a^2-b^2; and on dividing this by $(a+b)^2$ and a^2-b^2 we obtain the quotients $a-b$ and $a+b$, which have no common measure. Thus we obtain $12a^2b(a+b)^2(a-b)$ as the required L.C.M.

121. The following may be given as the *definition of the* L.C.M. *of two or more compound expressions.* Let two or more compound expressions contain powers of some common letter; then the expression of lowest dimensions in that letter which is measured by each of these expressions is called their least common multiple.

122. We shall now shew how to find the L.C.M. of two compound expressions. The demonstration however will not be fully understood at the present stage of the student's knowledge.

Let A and B denote the two expressions, and D their greatest common measure. Suppose $A=aD$, and $B=bD$. Then from the nature of the greatest common measure, a

and b have no common factor, and therefore their least common multiple is ab. Hence the expression of lowest dimensions which is measured by aD and bD is abD. And $abD = Ab = Ba = \frac{AB}{D}$.

Hence we have the following Rule for finding the L.C.M. of two compound expressions. *Divide the product of the expressions by their* G.C.M. Or we may give the rule thus: *Divide one of the expressions by their* G.C.M., *and multiply the quotient by the other expression.*

123. For example; required the L.C.M. of $x^2 - 4x + 3$ and $4x^3 - 9x^2 - 15x + 18$.

The G.C.M. is $x - 3$; see Art. 105. Divide $x^2 - 4x + 3$ by $x - 3$; the quotient is $x - 1$. Therefore the L.C.M. is $(x - 1)(4x^3 - 9x^2 - 15x + 18)$; and this gives, by multiplying out, $4x^4 - 13x^3 - 6x^2 + 33x - 18$.

It is however often convenient to have the L.C.M. expressed in factors, rather than multiplied out. We know that the G.C.M., which is $x - 3$, will measure the expression $4x^3 - 9x^2 - 15x + 18$; by division we obtain the quotient. Hence the L.C.M. is

$$(x - 3)(x - 1)(4x^2 + 3x - 6).$$

For another example, suppose we require the L.C.M. of $2x^2 - 7x + 5$ and $3x^2 - 7x + 4$.

The G.C.M. is $x - 1$: see Art. 111.

Also $\qquad (2x^2 - 7x + 5) \div (x - 1) = 2x - 5,$

and $\qquad (3x^2 - 7x + 4) \div (x - 1) = 3x - 4.$

Hence the L.C.M. is

$$(x - 1)(2x - 5)(3x - 4).$$

Again; required the L.C.M. of $2x^4 - 7x^3 - 4x^2 + x - 4$, and $3x^4 - 11x^3 - 2x^2 - 4x - 16$.

The G.C.M. is $x^2 - 3x - 4$: see Art. 111.

Also

$$(2x^4 - 7x^3 - 4x^2 + x - 4) \div (x^2 - 3x - 4) = 2x^2 - x + 1,$$

and

$$(3x^4 - 11x^3 - 2x^2 - 4x - 16) \div (x^2 - 3x - 4) = 3x^2 - 2x + 4.$$

Hence the L.C.M. is

$$(x^2-3x-4)(2x^2-x+1)(3x^2-2x+4).$$

124. It is obvious that, *every multiple of a common multiple of two or more expressions is a common multiple of those expressions.*

125. *Every common multiple of two expressions is a multiple of their least common multiple.*

Let A and B denote the two expressions, M their L.C.M.; and let N denote any other common multiple. Suppose, if possible, that when N is divided by M there is a remainder R; let q denote the quotient. Thus $R=N-qM$. Now A and B measure M and N, and therefore they measure R (Art. 106). But by the nature of division R is of *lower* dimensions than M; and thus there is a common multiple of A and B which is of lower dimensions than their L.C.M. This is absurd. Therefore there can be no remainder R; that is, N is a multiple of M.

126. Suppose now that we require the L.C.M. of *three* compound expressions, A, B, C. Find the L.C.M. of any two of them, say of A and B; let M denote this L.C.M.; then the L.C.M. of M and C will be the required L.C.M. of A, B, and C.

For every common multiple of M and C is a common multiple of A, B, and C, by Art. 124. And every common multiple of A and B is a multiple of M, by Art. 125; hence every common multiple of M and C is a common multiple of A, B, and C. Therefore the L.C.M. of M and C is the L.C.M. of A, B, and C.

127. In a similar manner we may find the L.C.M. of four expressions.

128. The theories of the greatest common measure and of the least common multiple are not necessary for the subsequent Chapters of the present work, and any difficulties which the student may find in them may be postponed until he has read the Theory of Equations. The examples however attached to the preceding Chapter and to the present Chapter should be carefully worked, on account of the exercise which they afford in all the fundamental processes of Algebra.

Examples. XIII.

Find the least common multiple in the following examples:

1. $4a^2b, \quad 6ab^2.$

2. $12a^3b^2c, \quad 18ab^2c^3.$

3. $8a^2x^2y^3, \quad 12b^2x^3y^2.$

4. $(a-b)^2, \quad a^2-b^2.$

5. $4a(a+b), \quad 6b(a^3+b^3).$

6. $a^2-b^2, \quad a^3-b^3.$

7. $x^2-3x-4, \quad x^2-x-12.$

8. $x^3+5x^2+7x+2, \quad x^2+6x+8.$

9. $12x^2+5x-3, \quad 6x^3+x^2-x.$

10. $x^3-6x^2+11x-6, \quad x^3-9x^2+26x-24.$

11. $x^3-7x-6, \quad x^3+8x^2+17x+10.$

12. $x^4+x^3+2x^2+x+1, \quad x^4-1.$

13. $x^4-2x^3-3x^2+8x-4, \quad x^4-5x^3+20x-16.$

14. $x^4+a^2x^2+a^4, \quad x^4-ax^3-a^3x+a^4.$

15. $4a^2b^2c, \quad 6ab^3c^2, \quad 18a^2bc^3.$

16. $8(a^2-b^2), \quad 12(a+b)^2, \quad 20(a-b)^2.$

17. $4(a+b), \quad 6(a^2-b^2), \quad 8(a^3+b^3).$

18. $15(a^2b-ab^2), \quad 21(a^3-ab^2), \quad 35(ab^2+b^3).$

19. $x^2-1, \quad x^3+1, \quad x^3-1.$

20. $x^2-1, \quad x^2+1, \quad x^4+1, \quad x^8-1.$

21. $x^2-1, \quad x^3+1, \quad x^3-1, \quad x^6+1.$

22. $x^2+3x+2, \quad x^2+4x+3, \quad x^2+5x+6.$

23. $x^2+2x-3, \quad x^3+3x^2-x-3, \quad x^3+4x^2+x-6.$

24. $x^2+5x+10, \quad x^3-19x-30, \quad x^3-15x-50.$

XIV. *Fractions.*

129. In this Chapter and the following four Chapters we shall treat of Fractions; and the student will find that the rules and demonstrations closely resemble those with which he is already familiar in Arithmetic.

130. By the expression $\frac{a}{b}$ we indicate that a unit is to be divided into b equal parts, and that a of such parts are to be taken. Here $\frac{a}{b}$ is called a *fraction*; a is called the *numerator*, and b is called the *denominator*. Thus the denominator indicates into how many equal parts the unit is to be divided, and the numerator indicates how many of those parts are to be taken.

Every integer or integral expression may be considered as a fraction with unity for its denominator; that is, for example, $a = \frac{a}{1}$, $b + c = \frac{b+c}{1}$.

131. In Algebra, as in Arithmetic, it is usual to give the following Rule for expressing a fraction as a mixed quantity: *Divide the numerator by the denominator, as far as possible, and annex to the quotient a fraction having the remainder for numerator, and the divisor for denominator.*

Examples. $\frac{24a}{7} = 3a + \frac{3a}{7}$.

$$\frac{a^2 + 3ab}{a + b} = a + \frac{2ab}{a + b}.$$

$$\frac{x^3 - 6x + 14}{x^2 - 3x + 4} = x + 3 + \frac{-x + 2}{x^2 - 3x + 4}$$

$$\text{or } = x + 3 - \frac{x - 2}{x^2 - 3x + 4}.$$

The student is recommended to pay *particular attention* to the last step; it is really an example of the use of brackets, namely, $+(-x+2)=-(x-2)$.

132. Rule for multiplying a fraction by an integer. *Either multiply the numerator by that integer, or divide the denominator by that integer.*

Let $\frac{a}{b}$ denote any fraction, and c any integer; then will $\frac{a}{b}\times c=\frac{ac}{b}$. For in each of the fractions $\frac{a}{b}$ and $\frac{ac}{b}$ the unit is divided into b equal parts, and c times as many parts are taken in $\frac{ac}{b}$ as in $\frac{a}{b}$; hence $\frac{ac}{b}$ is c times $\frac{a}{b}$.

This demonstrates the first form of the Rule.

Again; let $\frac{a}{bc}$ denote any fraction, and c any integer; then will $\frac{a}{bc}\times c=\frac{a}{b}$. For in each of the fractions $\frac{a}{bc}$ and $\frac{a}{b}$ the same number of parts is taken, but each part in $\frac{a}{b}$ is c times as large as each part in $\frac{a}{bc}$, because in $\frac{a}{bc}$ the unit is divided into c times as many parts as in $\frac{a}{b}$; hence $\frac{a}{b}$ is c times $\frac{a}{bc}$.

This demonstrates the second form of the Rule.

133. Rule for dividing a fraction by an integer. *Either multiply the denominator by that integer, or divide the numerator by that integer.*

Let $\frac{a}{b}$ denote any fraction, and c any integer; then will $\frac{a}{b}\div c=\frac{a}{bc}$. For $\frac{a}{b}$ is c times $\frac{a}{bc}$, by Art. 132; and therefore $\frac{a}{bc}$ is $\frac{1}{c}$th of $\frac{a}{b}$.

This demonstrates the first form of the Rule.

Again; let $\frac{ac}{b}$ denote any fraction, and c any integer; then will $\frac{ac}{b} \div c = \frac{a}{b}$. For $\frac{ac}{b}$ is c times $\frac{a}{b}$, by Art. 132; and therefore $\frac{a}{b}$ is $\frac{1}{c}$th of $\frac{ac}{b}$.

This demonstrates the second form of the Rule.

134. *If the numerator and denominator of any fraction be multiplied by the same integer, the value of the fraction is not altered.*

For if the numerator of a fraction be multiplied by any integer, the fraction will be *multiplied* by that integer; and the result will be *divided* by that integer if its denominator be multiplied by that integer. But if we multiply any number by an integer, and then divide the result by the same integer, the number is not altered.

The result may also be stated thus: if the numerator and denominator of any fraction be *divided* by the same integer, the value of the fraction is not altered.

Both these verbal statements are included in the algebraical statement $\frac{a}{b} = \frac{ac}{bc}$.

This result is of very great importance; many of the operations in Fractions depend on it, as we shall see in the next two Chapters.

135. The demonstrations given in this Chapter are satisfactory only when every letter denotes some *positive whole number;* but the results are *assumed* to be true whatever the letters denote. For the grounds of this assumption the student may hereafter consult the larger Algebra. The result contained in Art. 134 is the most important; the student will therefore observe that henceforth we assume that it is *always* true in Algebra that $\frac{a}{b} = \frac{ac}{bc}$, whatever a, b, and c may denote.

For example, if we put -1 for c we have $\frac{a}{b} = \frac{-a}{-b}$.

So also

$$\frac{a}{-b}=\frac{-a}{b}; \quad +\frac{a}{-b}=+\frac{-a}{b}=-\frac{a}{b}; \quad -\frac{a}{-b}=-\frac{-a}{b}=\frac{a}{b}.$$

In like manner, by assuming that $\frac{a}{b}\times c$ is *always* equal to $\frac{ac}{b}$ we obtain such results as the following:

$$\frac{a}{b}\times -1=\frac{-a}{b}=-\frac{a}{b}, \qquad \frac{a}{b}\times -2=\frac{-2a}{b}=-\frac{2a}{b}.$$

Examples. XIV.

Express the following fractions as mixed quantities:

1. $\frac{25x}{7}$. 2. $\frac{36ac+4c}{9}$. 3. $\frac{8a^2+3b}{4a}$. 4. $\frac{12x^2-5y}{6x}$.

5. $\frac{x^2+3x+2}{x+3}$. 6. $\frac{2x^2-6x-1}{x-3}$.

7. $\frac{x^3+ax^2-3a^2x-3a^3}{x-2a}$. 8. $\frac{x^3-2x^2}{x^2-x+1}$.

9. $\frac{x^4+1}{x-1}$. 10. $\frac{x^4-1}{x+1}$.

Multiply

11. $\frac{4a^2}{9b^2}$ by $3b$. 12. $\frac{8(a^2+b^2)}{9(a^2-b^2)}$ by $3(a-b)$.

13. $\frac{3(a-b)}{8(a^3+b^3)}$ by $4(a^2-ab+b^2)$. 14. $\frac{x^2}{(x^2-1)^2}$ by $x+1$.

Divide

15. $\frac{8x^2}{3y}$ by $2x$. 16. $\frac{9a^2-4b^2}{a+b}$ by $3a-2b$.

17. $\frac{10(a^3-b^3)}{3(a+b)}$ by $5(a^2+ab+b^2)$.

18. $\frac{x^6-1}{x^2+1}$ by x^2-x+1.

XV. *Reduction of Fractions.*

136. The result contained in Art. 134 will now be applied to two important operations, the reduction of a fraction to its lowest terms, and the reduction of fractions to a common denominator.

137. Rule for reducing a fraction to its lowest terms. *Divide the numerator and denominator of the fraction by their greatest common measure.*

For example; reduce $\frac{16a^4b^2c}{20a^3b^3d}$ to its lowest terms.

The G.C.M. of the numerator and the denominator is $4a^3b^2$; dividing both numerator and denominator by $4a^3b^2$, we obtain for the required result $\frac{4ac}{5bd}$. That is, $\frac{4ac}{5bd}$ is equal to $\frac{16a^4b^2c}{20a^3b^3d}$, but it is expressed in a more simple form; and it is said to be in the *lowest terms*, because it cannot be further simplified by the aid of Art. 134.

Again; reduce $\frac{x^2-4x+3}{4x^3-9x^2-15x+18}$ to its lowest terms.

The G.C.M. of the numerator and the denominator is $x-3$; dividing both numerator and denominator by $x-3$ we obtain for the required result $\frac{x-1}{4x^2+3x-6}$.

In some examples we may perceive that the numerator and denominator have a common factor, without using the rule for finding the G.C.M. Thus, for example,

$$\frac{(a-b)^2-c^2}{a^2-(b+c)^2}=\frac{(a-b+c)(a-b-c)}{(a+b+c)(a-b-c)}=\frac{a-b+c}{a+b+c}.$$

138. Rule for reducing fractions to a common denominator. *Multiply the numerator of each fraction by all the denominators except its own, for the numerator corresponding to that fraction; and multiply all the denominators together for the common denominator.*

For example; reduce $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ to a common denominator.

$$\frac{a}{b}=\frac{adf}{bdf}, \quad \frac{c}{d}=\frac{cbf}{dbf}, \quad \frac{e}{f}=\frac{ebd}{fbd}.$$

Thus $\frac{adf}{bdf}$, $\frac{cbf}{dbf}$, and $\frac{ebd}{fbd}$ are fractions of the same value respectively as $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$; and they have the common denominator bdf.

The Rule given in this Article will always reduce fractions to *a* common denominator, but not always to the *lowest* common denominator; it is therefore often convenient to employ another Rule which we shall now give.

139. Rule for reducing fractions to their lowest common denominator. *Find the least common multiple of the denominators, and take this for the common denominator; then for the new numerator corresponding to any of the proposed fractions, multiply the numerator of that fraction by the quotient which is obtained by dividing the least common multiple by the denominator of that fraction.*

For example; reduce $\frac{a}{yz}$, $\frac{b}{zx}$, $\frac{c}{xy}$ to the lowest common denominator. The least common multiple of the denominators is xyz; and

$$\frac{a}{yz}=\frac{ax}{xyz}, \quad \frac{b}{zx}=\frac{by}{xyz}, \quad \frac{c}{xy}=\frac{cz}{xyz}$$

EXAMPLES. XV.

Reduce the following fractions to their lowest terms:

1. $\dfrac{12a^4b^2x}{18a^2b^2y}$.

2. $\dfrac{a^2+ab}{2ab}$.

3. $\dfrac{a^2+ab}{a^2-ab}$.

4. $\dfrac{10a^2x}{5a^2x-15ay^2}$.

5. $\dfrac{4(a+b)^2}{5(a^2-b^2)}$.

6. $\dfrac{a^3+b^3}{a^2-b^2}$.

7. $\dfrac{x^2+3x+2}{x^2+6x+5}$.

8. $\dfrac{x^2+10x+21}{x^2-2x-15}$.

9. $\dfrac{2x^2+x-15}{2x^2-19x+35}$.

10. $\dfrac{x^2+(a+b)x+ab}{x^2+(a+c)x+ac}$.

11. $\dfrac{x^2-(a+b)x+ab}{x^2+(c-a)x-ac}$.

12. $\dfrac{3x^2+23x-36}{4x^2+33x-27}$.

13. $\dfrac{(x+a)^2-(b+c)^2}{(x+b)^2-(a+c)^2}$.

14. $\dfrac{x^2+5x+6}{x^3+x+10}$.

15. $\dfrac{x^2-10x+21}{x^3-46x-21}$.

16. $\dfrac{x^2+9x+20}{x^3+7x^2+14x+8}$.

17. $\dfrac{x^2+x-42}{x^3-10x^2+21x+18}$.

18. $\dfrac{6x^2-11x+5}{3x^3-2x^2-1}$.

19. $\dfrac{20x^2+x-12}{12x^3-5x^2+5x-6}$.

20. $\dfrac{x^2-2ax+a^2}{x^3-2ax^2+2a^2x-a^3}$.

21. $\dfrac{2x^3-5x^2-8x-16}{2x^3+11x^2+16x+16}$.

22. $\dfrac{x^3-3a^2x+2a^3}{2x^3+ax^2+a^2x-4a^3}$.

23. $\dfrac{x^3-8x-3}{x^4-7x^2+1}$.

24. $\dfrac{x^3+a^3}{x^4+a^2x^2+a^4}$.

25. $\dfrac{x^3-x^2-7x+3}{x^4+2x^3+2x-1}$.

26. $\dfrac{3x^4-14x^3-9x+2}{2x^4-9x^3-14x+3}$.

27. $\dfrac{3x^5 - 75a^4x}{2x^4 + 13a^2x^2 + 15a^4}$.

28. $\dfrac{x^4 - 1}{x^6 - 1}$.

29. $\dfrac{x^4 + x^3 + x^2 + x + 1}{x^5 - 1}$.

30. $\dfrac{x^6 + a^2x^3y}{x^6 - a^4y^2}$.

31. $\dfrac{x^4 + a^2x^2 + a^4}{x^6 - a^6}$.

32. $\dfrac{x^{m-1}y^{2n}}{x^{2m}y^{n+1}}$.

Reduce the following fractions to their lowest common denominator:

33. $\dfrac{3}{4x}, \dfrac{4}{6x^2}, \dfrac{5}{12x^3}$.

34. $\dfrac{1}{x+1}, \dfrac{3}{4x+4}, \dfrac{x}{x^2-1}$.

35. $\dfrac{a}{x-a}, \dfrac{x}{a-x}, \dfrac{a^3}{x^2-a^2}, \dfrac{ax}{a^2-x^2}$.

36. $\dfrac{a}{a-b}, \dfrac{b}{a+b}, \dfrac{ab}{a^2-b^2}, \dfrac{b^2}{a^2+b^2}$.

37. $\dfrac{1}{x-1}, \dfrac{x}{(x-1)^2}, \dfrac{3}{x+1}, \dfrac{4}{(x+1)^2}, \dfrac{5}{x^2-1}$.

38. $\dfrac{a}{x-a}, \dfrac{a+x}{x^2+ax+a^2}, \dfrac{ax}{x^3-a^3}$.

39. $\dfrac{1}{x^2-ax+a^2}, \dfrac{1}{x^2+ax+a^2}, \dfrac{a^2}{x^4+a^2x^2+a^4}$.

40. $\dfrac{1}{x^2-(a+b)x+ab}, \dfrac{1}{x^2-(a+c)x+ac}$,

$$\dfrac{1}{x^2-(b+c)x+bc}.$$

XVI. *Addition or Subtraction of Fractions.*

140. Rule for the addition or subtraction of fractions. *Reduce the fractions to a common denominator, then add or subtract the numerators and retain the common denominator.*

Examples. Add $\frac{a+c}{b}$ to $\frac{a-c}{b}$.

Here the fractions have already a common denominator, and therefore do not require reducing;

$$\frac{a+c}{b}+\frac{a-c}{b}=\frac{a+c+a-c}{b}=\frac{2a}{b}.$$

From $\frac{4a-3b}{c}$ take $\frac{3a-4b}{c}$.

$$\frac{4a-3b}{c}-\frac{3a-4b}{c}=\frac{4a-3b-(3a-4b)}{c}$$

$$=\frac{4a-3b-3a+4b}{c}=\frac{a+b}{c}.$$

The student is recommended to put down the work *at full*, as we have done in this example, in order to ensure accuracy.

Add $\frac{c}{a+b}$ to $\frac{c}{a-b}$.

Here the common denominator will be the product of $a+b$ and $a-b$, that is a^2-b^2.

$$\frac{c}{a+b}=\frac{c(a-b)}{a^2-b^2};\quad \frac{c}{a-b}=\frac{c(a+b)}{a^2-b^2}.$$

Therefore $\frac{c}{a+b}+\frac{c}{a-b}=\frac{c(a-b)+c(a+b)}{a^2-b^2}$

$$=\frac{ca-cb+ca+cb}{a^2-b^2}=\frac{2ca}{a^2-b^2}.$$

From $\frac{a+b}{a-b}$ take $\frac{a-b}{a+b}$.

The common denominator is a^2-b^2.

$$\frac{a+b}{a-b}=\frac{(a+b)^2}{a^2-b^2};\quad \frac{a-b}{a+b}=\frac{(a-b)^2}{a^2-b^2}.$$

Therefore $\frac{a+b}{a-b}-\frac{a-b}{a+b}=\frac{(a+b)^2-(a-b)^2}{a^2-b^2}$

$$=\frac{a^2+2ab+b^2-(a^2-2ab+b^2)}{a^2-b^2}=\frac{4ab}{a^2-b^2}.$$

From $\frac{x+1}{x^2-4x+3}$ take $\frac{4x^2-3x+2}{4x^3-9x^2-15x+18}$.

By Art. 123 the L.C.M. of the denominators is

$$(x-1)(x-3)(4x^2+3x-6);$$

$$\frac{x+1}{x^2-4x+3}=\frac{(x+1)(4x^2+3x-6)}{(x-1)(x-3)(4x^2+3x-6)},$$

$$\frac{4x^2-3x+2}{4x^3-9x^2-15x+18}=\frac{(4x^2-3x+2)(x-1)}{(x-1)(x-3)(4x^2+3x-6)}.$$

Therefore $\frac{x+1}{x^2-4x+3}-\frac{4x^2-3x+2}{4x^3-9x^2-15x+18}$

$$=\frac{(x+1)(4x^2+3x-6)-(4x^2-3x+2)(x-1)}{(x-1)(x-3)(4x^2+3x-6)}$$

$$=\frac{4x^3+7x^2-3x-6-(4x^3-7x^2+5x-2)}{(x-1)(x-3)(4x^2+3x-6)}$$

$$=\frac{14x^2-8x-4}{(x-1)(x-3)(4x^2+3x-6)}.$$

141. We have sometimes to *reduce a mixed quantity to a fraction;* this is a simple case of addition or subtraction of fractions.

Examples. $a+\frac{b}{c}=\frac{a}{1}+\frac{b}{c}=\frac{ac}{c}+\frac{b}{c}=\frac{ac+b}{c}$.

$$a+\frac{2ab}{a+b}=\frac{a}{1}+\frac{2ab}{a+b}=\frac{a(a+b)}{a+b}+\frac{2ab}{a+b}=\frac{a^2+3ab}{a+b}.$$

$$x+3-\frac{x-2}{x^2-3x+4}=\frac{x+3}{1}-\frac{x-2}{x^2-3x+4}$$

$$=\frac{(x+3)(x^2-3x+4)}{x^2-3x+4}-\frac{x-2}{x^2-3x+4}$$

$$=\frac{x^3-5x+12-(x-2)}{x^2-3x+4}=\frac{x^3-5x+12-x+2}{x^2-3x+4}=\frac{x^3-6x+14}{x^2-3x+4}.$$

142. Expressions may occur involving both addition and subtraction. Thus, for example, simplify

$$\frac{a}{a+b}+\frac{ab}{a^2-b^2}-\frac{a^2}{a^2+b^2}.$$

The L.C.M. of the denominators is $(a^2-b^2)(a^2+b^2)$, that is a^4-b^4.

$$\frac{a}{a+b}=\frac{a(a-b)(a^2+b^2)}{a^4-b^4}=\frac{a^4-a^3b+a^2b^2-ab^3}{a^4-b^4},$$

$$\frac{ab}{a^2-b^2}=\frac{ab(a^2+b^2)}{a^4-b^4}=\frac{a^3b+ab^3}{a^4-b^4},$$

$$\frac{a^2}{a^2+b^2}=\frac{a^2(a^2-b^2)}{a^4-b^4}=\frac{a^4-a^2b^2}{a^4-b^4}.$$

Therefore $\frac{a}{a+b}+\frac{ab}{a^2-b^2}-\frac{a^2}{a^2+b^2}$

$$=\frac{a^4-a^3b+a^2b^2-ab^3+a^3b+ab^3-(a^4-a^2b^2)}{a^4-b^4}$$

$$=\frac{a^4-a^3b+a^2b^2-ab^3+a^3b+ab^3-a^4+a^2b^2}{a^4-b^4}=\frac{2a^2b^2}{a^4-b^4}$$

Simplify $\frac{a}{(a-b)(a-c)}+\frac{b}{(b-c)(b-a)}+\frac{c}{(c-a)(c-b)}$.

The beginner should pay particular attention to this example. He is very liable to take the product of the denominators for the common denominator, and thus to render the operations extremely laborious.

The second fraction contains the factor $b-a$ in its denominator, and this factor differs from the factor $a-b$, which occurs in the denominator of the first fraction, only in the sign of each term; and by Art. 135,

$$\frac{b}{(b-c)(b-a)}=-\frac{b}{(b-c)(a-b)}.$$

Also the denominator of the third fraction can be put in a form which is more convenient for our object; for by the *Rule of Signs* we have

$$(c-a)(c-b)=(a-c)(b-c).$$

Hence the proposed expression may be put in the form

$$\frac{a}{(a-b)(a-c)}-\frac{b}{(b-c)(a-b)}+\frac{c}{(a-c)(b-c)};$$

and in this form we see at once that the L.C.M. of the denominators is $(a-b)(a-c)(b-c)$.

By reducing the fractions to the lowest common denominator the proposed expression becomes

$$\frac{a(b-c)-b(a-c)+c(a-b)}{(a-b)(a-c)(b-c)},$$

that is $\frac{ab-ac-ab+bc+ac-bc}{(a-b)(a-c)(b-c)}$, that is 0.

143. In this Chapter we have shewn how to combine two or more fractions into a single fraction; on the other hand we may, if we please, break up a single fraction into two or more fractions. For example,

$$\frac{3bc-4ac+5ab}{abc}=\frac{3bc}{abc}-\frac{4ac}{abc}+\frac{5ab}{abc}=\frac{3}{a}-\frac{4}{b}+\frac{5}{c}.$$

Examples. XVI.

Find the value of

1. $\frac{3a-5b}{4}+\frac{2a-b-c}{3}+\frac{a+b+c}{12}$.

2. $\frac{1}{a-b}+\frac{1}{a+b}$.

3. $\frac{a}{a-b}+\frac{b}{a+b}$.

4. $\frac{c}{a-b}-\frac{c}{a+b}$.

5. $\frac{1}{bc}+\frac{1}{ac}+\frac{1}{ab}$.

6. $\frac{1}{x+y}+\frac{2y}{x^2-y^2}$.

7. $\frac{1+3x}{1-3x}-\frac{1-3x}{1+3x}$.

8. $\frac{a}{x(a-x)}-\frac{x}{a(a-x)}$.

9. $\frac{a}{2a-2b}-\frac{b}{2b-2a}$.

10. $\frac{a}{a-x}+\frac{3a}{a+x}-\frac{2ax}{a^2-x^2}$.

11. $\frac{a-2b}{3c}-\frac{b-3c}{2a}+\frac{4ab+3bc}{6ac}$.

12. $\frac{a-b}{b}+\frac{2a}{a-b}-\frac{a^3+a^2b}{a^2b-b^3}$.

13. $\frac{2b-a}{x-b}+\frac{b-2a}{x+b}+\frac{3x(a-b)}{x^2-b^2}$.

14. $\frac{3}{x}-\frac{5}{2x-1}-\frac{2x-7}{4x^2-1}$.

15. $\dfrac{1}{x-2}-\dfrac{3}{x+2}+\dfrac{2x}{(x+2)^2}$.

16. $\dfrac{1}{a-b}+\dfrac{1}{a+b}-\dfrac{a}{a^2-b^2}$.

17. $\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}-\dfrac{a^2-x^2}{a^2+x^2}$.

18. $\dfrac{1}{x+1}-\dfrac{2}{x+2}+\dfrac{1}{x+3}$.

19. $\dfrac{x}{x-1}-\dfrac{2x}{x+1}+\dfrac{x}{x-2}$.

20. $\dfrac{4x}{y}-\dfrac{x-y}{x+y}+\dfrac{x+y}{x-y}$.

21. $x-\dfrac{x^2}{x-1}-\dfrac{x}{x+1}$.

22. $x-\dfrac{x^2}{x+1}+\dfrac{x}{x-1}$.

23. $\dfrac{1}{x-a}+\dfrac{1}{x+a}-\dfrac{2}{x}$.

24. $\dfrac{a}{a-b}+\dfrac{a}{a+b}+\dfrac{4a^2b^2}{a^4-b^4}$.

25. $\dfrac{x^2}{x^2-1}+\dfrac{x}{x-1}+\dfrac{x}{x+1}$.

26. $\dfrac{a}{a-x}+\dfrac{3a}{a+x}-\dfrac{2ax}{a^2+x^2}$.

27. $\dfrac{3}{2x-4}-\dfrac{1}{x+2}-\dfrac{x+10}{2x^2+8}$.

28. $\dfrac{2}{x+4}-\dfrac{x-3}{x^2-4x+16}+\dfrac{x^2}{x^3+64}$.

29. $\dfrac{1}{x^2-a^2}+\dfrac{1}{(x+a)^2}-\dfrac{1}{(x-a)^2}$.

30. $\dfrac{x^2+ax+a^2}{x^3-a^3}-\dfrac{x^2-ax+a^2}{x^3+a^3}$.

31. $\dfrac{x^2+y^2}{xy}-\dfrac{x^2}{xy+y^2}-\dfrac{y^2}{x^2+xy}$.

32. $\dfrac{x^2-2x+3}{x^3+1}+\dfrac{x-2}{x^2-x+1}-\dfrac{1}{x+1}$.

33. $\dfrac{1}{(x-3)(x-4)}-\dfrac{2}{(x-2)(x-4)}+\dfrac{1}{(x-2)(x-3)}$.

34. $\dfrac{1}{x(x+1)}-\dfrac{2x-3}{x(x+1)(x+2)}+\dfrac{1}{x(x+2)}$.

35. $\dfrac{1-2x}{3(x^2-x+1)}+\dfrac{x+1}{2(x^2+1)}+\dfrac{1}{6(x+1)}$.

36. $\dfrac{x-y}{x^2-xy+y^2}+\dfrac{1}{x+y}+\dfrac{xy}{x^3+y^3}$.

37. $\dfrac{1}{x-y}+\dfrac{x-y}{x^2+xy+y^2}+\dfrac{xy-2x^2}{x^3-y^3}$.

38. $\dfrac{x+1}{x^2+x+1}+\dfrac{x-1}{x^2-x+1}+\dfrac{2}{x^4+x^2+1}$.

39. $\dfrac{a+b}{ax+by}+\dfrac{a-b}{ax-by}+\dfrac{2(a^2x+b^2y)}{a^2x^2+b^2y^2}$.

40. $\dfrac{2x}{x^4-x^2+1}-\dfrac{1}{x^2-x+1}+\dfrac{1}{x^2+x+1}$.

41. $\dfrac{1}{x^2-7x+12}+\dfrac{2}{x^2-4x+3}-\dfrac{3}{x^2-5x+4}$.

42. $\dfrac{1}{x+a}-\dfrac{1}{x-a}+\dfrac{4a}{x^2-a^2}-\dfrac{2a}{x^2+a^2}$.

43. $\dfrac{1}{a-b}-\dfrac{1}{a+b}-\dfrac{2b}{a^2+b^2}-\dfrac{4b^3}{a^4+b^4}$.

44. $\dfrac{1}{x-3a}-\dfrac{1}{x+3a}+\dfrac{3}{x+a}-\dfrac{3}{x-a}$.

45. $\dfrac{1}{a-2b}-\dfrac{4}{a-b}+\dfrac{6}{a}-\dfrac{4}{a+b}+\dfrac{1}{a+2b}$.

46. $\dfrac{c}{(x-a)(a-b)}+\dfrac{c}{(x-b)(b-a)}$.

47. $\dfrac{a}{(x-a)(a-b)}+\dfrac{b}{(x-b)(b-a)}$.

48. $\dfrac{a^2}{(x-a)(a-b)}+\dfrac{b^2}{(x-b)(b-a)}$.

49. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-a)(b-c)}$.

50. $\dfrac{b}{(a-b)(a-c)}+\dfrac{a}{(b-a)(b-c)}$.

51. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-a)(b-c)}+\dfrac{1}{(c-a)(c-b)}$.

52. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-a)(b-c)}-\dfrac{1}{abc}$.

53. $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-a)(b-c)}+\dfrac{c^2}{(c-a)(c-b)}$.

54. $\dfrac{1}{x^2-(a+b)x+ab}+\dfrac{1}{x^2-(a+c)x+ac}$
$+\dfrac{1}{x^2-(b+c)x+bc}$.

55. $\dfrac{x+c}{x^2-(a+b)x+ab}+\dfrac{x+b}{x^2-(a+c)x+ac}$
$+\dfrac{x+a}{x^2-(b+c)x+bc}$.

56. $\dfrac{1}{(a-b)(a-c)(x-a)}+\dfrac{1}{(b-a)(b-c)(x-b)}$
$+\dfrac{1}{(c-a)(c-b)(x-c)}$.

XVII. *Multiplication of Fractions.*

144. Rule for the multiplication of fractions. *Multiply together the numerators for a new numerator, and the denominators for a new denominator.*

145. The following is the usual demonstration of the Rule. Let $\frac{a}{b}$ and $\frac{c}{d}$ be two fractions which are to be multiplied together; put $\frac{a}{b}=x$, and $\frac{c}{d}=y$; therefore

$$a=bx, \text{ and } c=dy;$$

therefore $$ac=bdxy;$$

divide by bd, thus $$\frac{ac}{bd}=xy.$$

But $$xy=\frac{a}{b}\times\frac{c}{d};$$

therefore $$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}.$$

And ac is the product of the numerators, and bd the product of the denominators; this demonstrates the Rule.

Similarly the Rule may be demonstrated when more than two fractions are multiplied together.

146. We shall now give some examples. Before multiplying together the factors of the new numerator and the factors of the new denominator, it is advisable to examine if any factor occurs in both the numerator and denominator, as it may be struck out of both, and the result will thus be simplified; see Art. 137.

Multiply a by $\frac{b}{c}$.

$$a=\frac{a}{1};\ \frac{a}{1}\times\frac{b}{c}=\frac{ab}{c}.$$

Hence $a\frac{b}{c}$ and $\frac{ab}{c}$ are equivalent; so, for example, $4\frac{x}{5}=\frac{4x}{5}$; and $\frac{1}{4}(2x-3)=\frac{2x-3}{4}$.

Multiply $\frac{x}{y}$ by $\frac{x}{y}$.

$$\frac{x}{y} \times \frac{x}{y} = \frac{x \times x}{y \times y} = \frac{x^2}{y^2};$$

thus $$\left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2}.$$

Multiply $\frac{3a}{4b}$ by $\frac{8c}{9a}$.

$$\frac{3a}{4b} \times \frac{8c}{9a} = \frac{3a \times 8c}{4b \times 9a} = \frac{2c \times 12a}{3b \times 12a} = \frac{2c}{3b}.$$

Multiply $\frac{3a^2}{(a+b)^2}$ by $\frac{4(a^2-b^2)}{3ab}$.

$$\frac{3a^2}{(a+b)^2} \times \frac{4(a^2-b^2)}{3ab} = \frac{4a(a-b) \times 3a(a+b)}{b(a+b) \times 3a(a+b)} = \frac{4a(a-b)}{b(a+b)}.$$

Multiply $\frac{a}{b} + \frac{b}{a} + 1$ by $\frac{a}{b} + \frac{b}{a} - 1$.

$$\frac{a}{b} + \frac{b}{a} + 1 = \frac{a^2}{ab} + \frac{b^2}{ab} + \frac{ab}{ab} = \frac{a^2+b^2+ab}{ab},$$

$$\frac{a}{b} + \frac{b}{a} - 1 = \frac{a^2}{ab} + \frac{b^2}{ab} - \frac{ab}{ab} = \frac{a^2+b^2-ab}{ab};$$

$$\frac{a^2+b^2+ab}{ab} \times \frac{a^2+b^2-ab}{ab} = \frac{(a^2+b^2+ab)(a^2+b^2-ab)}{a^2b^2}$$

$$= \frac{(a^2+b^2)^2 - a^2b^2}{a^2b^2} = \frac{a^4+b^4+a^2b^2}{a^2b^2}.$$

Or we may proceed thus:

$$\left(\frac{a}{b} + \frac{b}{a} + 1\right)\left(\frac{a}{b} + \frac{b}{a} - 1\right) = \left(\frac{a}{b} + \frac{b}{a}\right)^2 - 1;$$

$$\left(\frac{a}{b} + \frac{b}{a}\right)^2 = \left(\frac{a}{b}\right)^2 + 2\frac{a}{b}\frac{b}{a} + \left(\frac{b}{a}\right)^2 = \frac{a^2}{b^2} + 2 + \frac{b^2}{a^2};$$

therefore

$$\left(\frac{a}{b}+\frac{b}{a}+1\right)\left(\frac{a}{b}+\frac{b}{a}-1\right)=\frac{a^2}{b^2}+2+\frac{b^2}{a^2}-1=\frac{a^2}{b^2}+\frac{b^2}{a^2}+1.$$

The two results agree, for $\frac{a^2}{b^2}+\frac{b^2}{a^2}+1=\frac{a^4+b^4+a^2b^2}{a^2b^2}$.

Multiply together $\frac{1-a^2}{b+b^2}$, $\frac{1-b^2}{a+a^2}$, and $b+\frac{ab}{1-a}$.

We might multiply together the first two factors, and then multiply the product separately by b and by $\frac{ab}{1-a}$, and add the results; but it is more convenient to reduce the *mixed quantity* $b+\frac{ab}{1-a}$ to a single fraction. Thus

$$b+\frac{ab}{1-a}=\frac{b(1-a)+ab}{1-a}=\frac{b}{1-a}.$$

Then

$$\frac{1-a^2}{b+b^2}\times\frac{1-b^2}{a+a^2}\times\frac{b}{1-a}=\frac{(1-a^2)(1-b^2)b}{b(1+b)a(1+a)(1-a)}=\frac{1-b}{a}.$$

147. As we have already done in former Chapters, we must here give some results which the student must *assume* to be capable of explanation, and which he must use as rules in working examples which may be proposed. See Arts. 63 and 135.

Multiply $\frac{a}{b}$ by $-\frac{c}{d}$.

$$\frac{a}{b}\times-\frac{c}{d}=\frac{a}{b}\times\frac{-c}{d}=\frac{-ac}{bd}=-\frac{ac}{bd}.$$

Multiply $-\frac{a}{b}$ by $\frac{c}{d}$.

$$-\frac{a}{b}\times\frac{c}{d}=\frac{-a}{b}\times\frac{c}{d}=\frac{-ac}{bd}=-\frac{ac}{bd}.$$

Multiply $-\frac{a}{b}$ by $-\frac{c}{d}$.

$$-\frac{a}{b}\times-\frac{c}{d}=\frac{-a}{b}\times\frac{-c}{d}=\frac{ac}{bd}.$$

Examples. XVII.

Find the value of the following:

1. $\dfrac{2a}{3b} \times \dfrac{6bc}{5a^2}$.

2. $\dfrac{a^2}{bc} \times \dfrac{b^2}{ac} \times \dfrac{c^2}{ab}$.

3. $\dfrac{a^2b}{x^2y} \times \dfrac{b^2c}{y^2z} \times \dfrac{c^2a}{z^2x}$.

4. $\dfrac{x+1}{x-1} \times \dfrac{x+2}{x^2-1} \times \dfrac{x-1}{(x+2)^2}$.

5. $\dfrac{xa}{x+a} \times \left(\dfrac{x}{a} - \dfrac{a}{x}\right)$.

6. $\left(b+\dfrac{a^2}{b}\right)\left(a-\dfrac{b^2}{a}\right)$.

7. $\left(a+\dfrac{ab}{a-b}\right)\left(b-\dfrac{ab}{a+b}\right)$.

8. $\dfrac{x(a-x)}{a^2+2ax+x^2} \times \dfrac{a(a+x)}{a^2-2ax+x^2}$.

9. $\dfrac{x^6-y^6}{x^4+2x^2y^2+y^4} \times \dfrac{x^2+y^2}{x^2-xy+y^2} \times \dfrac{x+y}{x^3-y^3}$.

10. $\dfrac{x^2-(a+b)x+ab}{x^2-(a+c)x+ac} \times \dfrac{x^2-c^2}{x^2-b^2}$.

11. $\dfrac{x^2+xy}{x^2+y^2} \times \left(\dfrac{x}{x-y} - \dfrac{y}{x+y}\right)$.

12. $\left(\dfrac{a}{bc} - \dfrac{b}{ac} - \dfrac{c}{ab} - \dfrac{2}{a}\right) \times \left(1 - \dfrac{2c}{a+b+c}\right)$.

13. $\left(\dfrac{x^2}{a^2} + \dfrac{a^2}{x^2} - \dfrac{x}{a} - \dfrac{a}{x} + 1\right) \times \left(\dfrac{x}{a} - \dfrac{a}{x}\right)$.

14. $\left(\dfrac{x}{a} - \dfrac{a}{x} + \dfrac{y}{b} - \dfrac{b}{y}\right) \times \left(\dfrac{x}{a} - \dfrac{a}{x} - \dfrac{y}{b} + \dfrac{b}{y}\right)$.

15. $\dfrac{x^2-2x+1}{x^2-5x+6} \times \dfrac{x^2-4x+4}{x^2-4x+3} \times \dfrac{x^2-6x+9}{x^2-3x+2}$.

XVIII. *Division of Fractions.*

148. Rule for dividing one fraction by another. *Invert the divisor and proceed as in Multiplication.*

149. The following is the usual demonstration of the Rule. Suppose we have to divide $\frac{a}{b}$ by $\frac{c}{d}$; put $\frac{a}{b}=x$, and $\frac{c}{d}=y$; therefore

$$a=bx, \text{ and } c=dy;$$

therefore $$ad=bdx, \text{ and } bc=bdy;$$

therefore $$\frac{ad}{bc}=\frac{bdx}{bdy}=\frac{x}{y}.$$

But $$\frac{x}{y}=x\div y=\frac{a}{b}\div\frac{c}{d};$$

therefore $$\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}=\frac{a}{b}\times\frac{d}{c}.$$

150. We shall now give some examples.

Divide a by $\frac{b}{c}$.

$$a=\frac{a}{1}; \qquad \frac{a}{1}\div\frac{b}{c}=\frac{a}{1}\times\frac{c}{b}=\frac{ac}{b}.$$

Divide $\frac{3a}{4b}$ by $\frac{9a}{8c}$.

$$\frac{3a}{4b}\div\frac{9a}{8c}=\frac{3a}{4b}\times\frac{8c}{9a}=\frac{2c\times 12a}{3b\times 12a}=\frac{2c}{3b}.$$

Divide $\frac{ab-b^2}{(a+b)^2}$ by $\frac{b^2}{a^2-b^2}$.

$$\frac{ab-b^2}{(a+b)^2}\div\frac{b^2}{a^2-b^2}=\frac{ab-b^2}{(a+b)^2}\times\frac{a^2-b^2}{b^2}$$

$$=\frac{b(a-b)(a+b)(a-b)}{b^2(a+b)^2}=\frac{(a-b)^2}{b(a+b)}.$$

151. Complex fractional expressions may be simplified by the aid of some or all of the rules respecting fractions which have now been given. The following are examples.

Simplify $\left\{\frac{a+b}{a-b}+\frac{a-b}{a+b}\right\}\div\left\{\frac{a+b}{a-b}-\frac{a-b}{a+b}\right\}$.

$$\frac{a+b}{a-b}+\frac{a-b}{a+b}=\frac{(a+b)^2+(a-b)^2}{(a-b)(a+b)}=\frac{2a^2+2b^2}{a^2-b^2},$$

$$\frac{a+b}{a-b}-\frac{a-b}{a+b}=\frac{(a+b)^2-(a-b)^2}{(a-b)(a+b)}=\frac{4ab}{a^2-b^2},$$

$$\frac{2a^2+2b^2}{a^2-b^2}\div\frac{4ab}{a^2-b^2}=\frac{2a^2+2b^2}{a^2-b^2}\times\frac{a^2-b^2}{4ab}=\frac{a^2+b^2}{2ab}.$$

In this example the factors $a-b$ and $a+b$ *are multiplied together*, and the result a^2-b^2 is used instead of $(a+b)(a-b)$; in general however the student will find it advisable *not to multiply the factors* together in the course of the operation, because an opportunity may occur of striking out a common factor from the numerator and denominator of his result.

Simplify $\dfrac{1}{a+\dfrac{1}{1+\dfrac{a+1}{3-a}}}$.

$$1+\frac{a+1}{3-a}=\frac{3-a}{3-a}+\frac{a+1}{3-a}=\frac{3-a+a+1}{3-a}=\frac{4}{3-a},$$

$$1\div\frac{4}{3-a}=\frac{1}{1}\times\frac{3-a}{4}=\frac{3-a}{4},$$

$$a+\frac{3-a}{4}=\frac{4a}{4}+\frac{3-a}{4}=\frac{3+3a}{4},$$

$$1\div\frac{3+3a}{4}=\frac{1}{1}\times\frac{4}{3+3a}=\frac{4}{3+3a}.$$

Find the value of $\left(\frac{2x-a}{2x-b}\right)^2 - \frac{a-x}{b-x}$ when $x=\frac{ab}{a+b}$.

$$2x-a=\frac{2ab}{a+b}-\frac{a}{1}=\frac{2ab-a(a+b)}{a+b}=\frac{ab-a^2}{a+b};$$

$$2x-b=\frac{2ab}{a+b}-\frac{b}{1}=\frac{2ab-b(a+b)}{a+b}=\frac{ab-b^2}{a+b}.$$

Therefore $\frac{2x-a}{2x-b}=\frac{ab-a^2}{a+b}\div\frac{ab-b^2}{a+b}=\frac{ab-a^2}{a+b}\times\frac{a+b}{ab-b^2}$

$$=\frac{ab-a^2}{ab-b^2}=\frac{a(b-a)}{b(a-b)}=-\frac{a}{b};$$

therefore $\left(\frac{2x-a}{2x-b}\right)^2=\left(-\frac{a}{b}\right)^2=\frac{a^2}{b^2}.$

Again, $a-x=\frac{a}{1}-\frac{ab}{a+b}=\frac{a(a+b)-ab}{a+b}=\frac{a^2}{a+b};$

$$b-x=\frac{b}{1}-\frac{ab}{a+b}=\frac{b(a+b)-ab}{a+b}=\frac{b^2}{a+b}.$$

Therefore $\frac{a-x}{b-x}=\frac{a^2}{a+b}\div\frac{b^2}{a+b}=\frac{a^2}{a+b}\times\frac{a+b}{b^2}=\frac{a^2}{b^2}.$

Therefore $\left(\frac{2x-a}{2x-b}\right)^2-\frac{a-x}{b-x}=\frac{a^2}{b^2}-\frac{a^2}{b^2}=0.$

152. The results given in Art. 147 must be given again here in connexion with Division of Fractions.

Since $\frac{a}{b}\times-\frac{c}{d}=-\frac{ac}{bd}$, and $-\frac{a}{b}\times\frac{c}{d}=-\frac{ac}{bd}$;

we have $-\frac{ac}{bd}\div-\frac{c}{d}=\frac{a}{b}$, and $-\frac{ac}{bd}\div\frac{c}{d}=-\frac{a}{b}$.

Also since $-\frac{a}{b}\times-\frac{c}{d}=\frac{ac}{bd}$, we have

$$\frac{ac}{bd}\div-\frac{c}{d}=-\frac{a}{b}.$$

EXAMPLES. XVIII.

Divide

1. $\frac{4a^2b}{5x^2y}$ by $\frac{2ab^2}{15xy^2}$.

2. $\frac{3a^2b^3c^4}{4x^2y^3z^4}$ by $\frac{4a^4b^3c^2}{3x^4y^3z^2}$.

3. $\frac{1}{x^2-y^2}$ by $\frac{1}{x-y}$.

4. $\frac{6(ab-b^2)}{a(a+b)^2}$ by $\frac{2b^2}{a(a^2-b^2)}$.

5. $\frac{a^2-4x^2}{a^2+4ax}$ by $\frac{a^2-2ax}{ax+4x^2}$.

6. $\frac{8x^3}{x^3-y^3}$ by $\frac{4x^3}{x^2+xy+y^2}$.

7. $\frac{a^3+3a^2x+3ax^2+x^3}{x^3+y^3}$ by $\frac{(a+x)^2}{x^2-xy+y^2}$.

8. $\frac{x^2+(a+c)x+ac}{x^2+(b+c)x+bc}$ by $\frac{x^2-a^2}{x^2-b^2}$.

9. $\frac{a^2+b^2+2ab-c^2}{c^2-a^2-b^2+2ab}$ by $\frac{a+b+c}{b+c-a}$.

10. $\frac{x^2+xy+y^2}{x^3+y^3}$ by $\frac{x^3-y^3}{x^2-xy+y^2}$.

11. $\frac{x^2-3x+2}{x^2-6x+9}$ by $\frac{x^2-5x+6}{x^2-2x+1}$.

12. $\left(1+\frac{x}{y}\right)\left(1-\frac{x}{y}\right)$ by $\frac{y}{x^2+y^2}$.

13. $5x^2-\frac{1}{5}$ by $x+\frac{1}{5}$.

14. $a^3-\frac{1}{a^3}$ by $a-\frac{1}{a}$.

15. $\frac{x^4}{a^4}-\frac{a^4}{x^4}$ by $\frac{x}{a}-\frac{a}{x}$.

16. $\frac{x^2}{a} - 8a + \frac{12a^3}{x^2}$ by $x - \frac{2a^2}{x}$.

17. $\frac{x^2}{y^3} - \frac{1}{x}$ by $\frac{x}{y^2} + \frac{1}{y} + \frac{1}{x}$.

18. $\frac{x^2}{a^2} + 1 + \frac{a^2}{x^2}$ by $\frac{x}{a} - 1 + \frac{a}{x}$.

19. $1 + \left(\frac{a-x}{a+x}\right)^2$ by $1 - \left(\frac{a-x}{a+x}\right)^3$.

20. $\frac{x^3}{a^3} + \frac{a^3}{x^3} - 3\left(\frac{x^2}{a^2} - \frac{a^2}{x^2}\right) + \frac{x}{a} + \frac{a}{x}$ by $\frac{x}{a} + \frac{a}{x}$.

Simplify the following expressions:

21. $\dfrac{\frac{3x}{2} + \frac{x-1}{3}}{\frac{13}{6}(x+1) - \frac{x}{3} - 2\frac{1}{2}}$.

22. $\dfrac{x - 1 + \frac{6}{x-6}}{x - 2 + \frac{3}{x-6}}$.

23. $\frac{3}{x+1} - \dfrac{2x-1}{x^2 + \frac{x}{2} - \frac{1}{2}}$.

24. $\dfrac{x-a}{x - \frac{(x-b)(x-c)}{x+a}}$.

25. $1 - \dfrac{1}{1 + \frac{1}{x}}$.

26. $1 + \dfrac{x}{1 + x + \frac{2x^2}{1-x}}$.

27. $\dfrac{1}{1 - \dfrac{1}{1 + \frac{1}{x}}}$.

28. $\dfrac{1}{1 + \dfrac{x}{1 + x + \frac{2x^2}{1-x}}}$.

29. $\left(\frac{x}{x-y} - \frac{y}{x+y}\right) \div \left(\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2-y^2}\right)$.

30. $\left(\frac{2x}{x+y} + \frac{y}{x-y} - \frac{y^2}{x^2-y^2}\right) \div \left(\frac{1}{x+y} + \frac{x}{x^2-y^2}\right)$.

31. $\dfrac{x+\dfrac{1}{y}}{x+\dfrac{1}{y+\dfrac{1}{z}}}-\dfrac{1}{y(xyz+x+z)}$.

32. $\left(\dfrac{a-b}{a+b}+\dfrac{a+b}{a-b}\right)\div\left(\dfrac{a^2-b^2}{a^2+b^2}+\dfrac{a^2+b^2}{a^2-b^2}\right)$.

Find the values of the following expressions:

33. $\dfrac{a-x}{b-x}$ when $x=\dfrac{ab}{a+b}$.

34. $\dfrac{x-a}{b}-\dfrac{x-b}{a}$ when $x=\dfrac{a^2}{a-b}$.

35. $\dfrac{x}{a}+\dfrac{x}{b-a}-\dfrac{a}{a+b}$ when $x=\dfrac{a^2(b-a)}{b(b+a)}$.

36. $\dfrac{a^2x+b^2y}{x+y}$ when $a=\dfrac{2}{3}$ and $b=\dfrac{2}{3}$.

37. $\dfrac{x}{x+y}+\dfrac{y}{x-y}-\dfrac{y^2}{x^2-y^2}$ when $y=\dfrac{3x}{4}$.

38. $\dfrac{x+2a}{2b-x}+\dfrac{x-2a}{2b+x}-\dfrac{4ab}{4b^2-x^2}$ when $x=\dfrac{ab}{a+b}$.

39. $\left(\dfrac{x-a}{x-b}\right)^3-\dfrac{x-2a+b}{x+a-2b}$ when $x=\dfrac{a+b}{2}$.

40. $\dfrac{x+y-1}{x-y+1}$ when $x=\dfrac{a+1}{\text{[illegible]}}$, and $y=\dfrac{ab+a}{\text{[illegible]}}$.

XIX. *Simple Equations.*

153. When two algebraical expressions are connected by the sign of equality the whole is called an equation. The expressions thus connected are called *sides* of the equation or *members* of the equation. The expression to the left of the sign of equality is called the *first* side, and the expression to the right is called the *second* side.

154. An *identical equation* is one in which the two sides are equal whatever numbers the letters represent; for example, the following are identical equations,

$$(x+a)(x-a)=x^2-a^2,$$

$$(x+a)^2=x^2+2xa+a^2,$$

$$(x+a)(x^2-xa+a^2)=x^3+a^3;$$

that is, these algebraical statements are true whatever numbers x and a may represent. The student will see that up to the present point he has been almost exclusively occupied with results of this kind, that is, with identical equations.

An identical equation is called briefly an *identity.*

155. An *equation of condition* is one which is not true whatever numbers the letters represent, but only when the letters represent some particular number or numbers. For example, $x+1=7$ cannot be true unless $x=6$. An equation of condition is called briefly an *equation.*

156. A letter to which a particular value or values must be given in order that the statement contained in an equation may be true, is called an *unknown quantity.* Such particular value of the unknown quantity is said to *satisfy the equation*, and is called a *root of the equation.* To *solve* an equation is to find the root or roots.

157. An equation involving one unknown quantity is said to be of as many dimensions as the index of the highest power of the unknown quantity. Thus, if x denote

the unknown quantity, the equation is said to be of *one* dimension when x occurs only in the *first* power; such an equation is also called a *simple equation*, or an equation of the *first degree*. If x^2 occurs, and no higher power of x, the equation is said to be of *two* dimensions; such an equation is also called a *quadratic equation*, or an equation of the *second degree*. If x^3 occurs, and no higher power of x, the equation is said to be of *three* dimensions; such an equation is also called a *cubic equation*, or an equation of the *third degree*. And so on.

It must be observed that these definitions suppose both members of the equation to be *integral expressions so far as relates to* x.

158. In the present Chapter we shall shew how to solve simple equations. We have first to indicate some operations which may be performed on an equation without destroying the equality which it expresses.

159. *If every term on each side of an equation be multiplied by the same number the results are equal.*

The *truth* of this statement follows from the obvious principle, that if equals be multiplied by the same number the results are equal; and the *use* of this statement will be seen immediately.

Likewise *if every term on each side of an equation be divided by the same number the results are equal.*

160. The principal use of Art. 159 is to *clear an equation of fractions;* this is effected by multiplying every term by the product of all the denominators of the fractions, or, if we please, by the least common multiple of those denominators. Suppose, for example, that

$$\frac{x}{3}+\frac{x}{4}+\frac{x}{6}=9.$$

Multiply every term by $3 \times 4 \times 6$; thus

$$4 \times 6 \times x + 3 \times 6 \times x + 3 \times 4 \times x = 3 \times 4 \times 6 \times 9,$$

$$\text{that is,} \quad 24x + 18x + 12x = 648;$$

divide every term by 6; thus

$$4x + 3x + 2x = 108.$$

Instead of multiplying every term by $3 \times 4 \times 6$, we may multiply every term by 12, which is the L.C.M. of the denominators 3, 4, and 6; we should then obtain at once

$$4x + 3x + 2x = 108;$$

that is, $$9x = 108;$$

divide both sides by 9; therefore

$$x = \frac{108}{9} = 12.$$

Thus 12 is the *root* of the proposed equation. We may *verify* this by putting 12 for x in the original equation. The first side becomes

$$\frac{12}{3} + \frac{12}{4} + \frac{12}{6}, \text{ that is } 4 + 3 + 2, \text{ that is } 9;$$

which agrees with the second side.

161. *Any term may be transposed from one side of an equation to the other side by changing its sign.*

Suppose, for example, that $x - a = b - y$.

Add a to each side; then

$$x - a + a = b - y + a,$$

that is $$x = b - y + a.$$

Subtract b from each side; thus

$$x - b - b + a - y - b = a - y.$$

Here we see that $-a$ has been removed from one side of the equation, and appears as $+a$ on the other side; and $+b$ has been removed from one side and appears as $-b$ on the other side.

162. *If the sign of every term of an equation be changed the equality still holds.*

This follows from Art. 161, by transposing every term. Thus suppose, for example, that $x - a = b - y$.

By transposition $y-b=a-x$,

that is, $a-x=y-b$;

and this result is what we shall obtain if we change the sign of every term in the original equation.

163. We can now give a Rule for the solution of any simple equation with one unknown quantity. *Clear the equation of fractions, if necessary; transpose all the terms which involve the unknown quantity to one side of the equation, and the known quantities to the other side; divide both sides by the coefficient, or the sum of the coefficients, of the unknown quantity, and the root required is obtained.*

164. We shall now give some examples.

Solve $7x+25=35+5x$.

Here there are no fractions; by transposing we have

$$7x-5x=35-25;$$

that is, $2x=10$;

divide by 2; therefore $x=\frac{10}{2}=5$.

We may verify this result by putting 5 for x in the original equation; then each side is equal to 60.

165. Solve $4(3x-2)-2(4x-3)-3(4-x)=0$.

Perform the multiplications indicated; thus

$$12x-8-(8x-6)-(12-3x)=0.$$

Remove the brackets; thus

$$12x-8-8x+6-12+3x=0;$$

collect the terms, $7x-14=0$;

transpose, $7x=14$;

divide by 7, $x=\frac{14}{7}=2$.

The student will find it a useful exercise to verify the correctness of his solutions. Thus in the above example,

if we put 2 for x in the original equation we shall obtain $16-10-6$, that is 0, as it should be.

166. Solve $x-2-(2x-3)=\frac{3x+1}{2}$.

Remove the brackets; thus

$$x-2-2x+3=\frac{3x+1}{2},$$

that is, $1-x=\frac{3x+1}{2}$;

multiply by 2, $2-2x=3x+1$,

transpose, $2-1=2x+3x$;

that is, $1=5x$, or $5x=1$;

therefore $x=\frac{1}{5}$.

167. Solve $\frac{5x+4}{2}-\frac{7x+5}{10}=5\frac{3}{5}-\frac{x-1}{2}$.

$5\frac{3}{5}=\frac{28}{5}$; the L.C.M. of the denominators is 10; multiply by 10;

thus $5(5x+4)-(7x+5)=28\times2-5(x-1)$;

that is, $25x+20-7x-5=56-5x+5$;

transpose, $25x-7x+5x=56+5-20+5$;

that is, $23x=46$;

therefore $x=\frac{46}{23}=2$.

The beginner is recommended to put down all the work at full, as in this example, in order to ensure accuracy. Mistakes with respect to the signs are often made in clearing an equation of fractions. In the above equation the fraction $-\frac{7x+5}{10}$ has to be multiplied by 10, and it is advisable to put the result first in the form $-(7x+5)$, and afterwards in the form $-7x-5$, in order to secure attention to the signs.

168. Solve $\frac{1}{3}(5x+3)-\frac{1}{7}(16-5x)=37-4x$.

By Art. 146 this is the same as

$$\frac{5x+3}{3}-\frac{16-5x}{7}=37-4x.$$

Multiply by 21; thus $7(5x+3)-3(16-5x)=21(37-4x)$,

that is, $35x+21-48+15x=777-84x$;

transpose, $35x+15x+84x=777-21+48$;

that is, $134x=804$;

therefore $x=\frac{804}{134}=6$.

169. Solve $\frac{6x+15}{11}-\frac{8x-10}{7}=\frac{4x-7}{5}$.

Multiply by the product of 11, 7, and 5; thus

$$35(6x+15)-55(8x-10)=77(4x-7),$$

that is, $210x+525-440x+550=308x-539$;

transpose, $210x-440x-308x=-539-525-550$;

change the signs, $440x+308x-210x=539+525+550$,

that is, $538x=1614$;

therefore $x=\frac{1614}{538}=3$.

Examples. XIX.

1. $5x+50=4x+56$.
2. $16x-11=7x+70$.
3. $24x-49=19x-14$.
4. $3x+23=78-2x$.
5. $7(x-18)=3(x-14)$.
6. $16x=38-3(4-x)$.
7. $7(x-3)=9(x+1)-38$.
8. $5(x-7)+63=9x$.
9. $59(x-7)=61(9-x)-2$.
10. $72(x-5)=63(5-x)$.
11. $28(x+9)=27(46-x)$.
12. $x+\frac{x}{2}+\frac{x}{3}=11$.

13. $\frac{x}{3} - \frac{x}{4} + \frac{1}{6} = \frac{x}{8} + \frac{1}{12}$.

14. $\frac{4x}{3} + 24 = 2x + 6$.

15. $\frac{x}{5} + \frac{x}{3} = x - 7$.

16. $36 - \frac{4x}{9} = 8$.

17. $\frac{2x}{3} + 4 = \frac{7x}{12} + 9$.

18. $\frac{3x}{4} + 5 = \frac{5x}{6} + 2$.

19. $56 - \frac{3x}{4} = 48 - \frac{5x}{8}$.

20. $\frac{x}{6} - 4 = 24 - \frac{x}{8}$.

21. $\frac{2x}{3} + 12 = \frac{4x}{5} + 6$.

22. $\frac{2x}{3} = \frac{176 - 4x}{5}$.

23. $\frac{7x}{8} - 5 = \frac{9x}{10} - 8$.

24. $\frac{5x}{9} - 8 = 74 - \frac{7x}{12}$.

25. $\frac{3x}{4} + \frac{180 - 5x}{6} = 29$.

26. $\frac{x}{2} + \frac{x+1}{7} = x - 2$.

27. $4(x-3) - 7(x-4) = 6 - x$.

28. $\frac{x}{3} - \frac{1}{3} - \frac{x}{4} + \frac{1}{4} = \frac{x}{5} - \frac{1}{5} - \frac{x}{6} + \frac{1}{6}$.

29. $1 + \frac{x}{2} - \frac{2x}{3} = \frac{3x}{4} - 4\frac{1}{2}$.

30. $2x - \frac{19 - 2x}{2} = \frac{2x - 11}{2}$.

31. $\frac{x+1}{3} - \frac{3x-1}{5} = x - 2$.

32. $x + \frac{3x-9}{5} = 4 - \frac{5x-12}{3}$.

33. $\frac{10x+3}{3} - \frac{6x-7}{2} = 10x - 10$.

34. $\frac{5x-7}{2} - \frac{2x+7}{3} = 3x - 14$.

35. $x - 1 - \frac{x-2}{2} + \frac{x-3}{3} = 0$.

36. $\frac{x+3}{2}+\frac{x+4}{3}+\frac{x+5}{4}=16.$

37. $\frac{7x+9}{4}=7+x-\frac{2x-1}{9}.$

38. $\frac{3x-4}{2}-\frac{6x-5}{8}=\frac{3x-1}{16}.$

39. $\frac{2x-5}{3}-\frac{5x-3}{4}+2\frac{2}{3}=0.$

40. $\frac{x-3}{4}=\frac{x-5}{6}+\frac{x-1}{9}.$

41. $\frac{x-1}{2}-\frac{x-3}{4}+\frac{x-5}{6}=4.$

42. $\frac{x}{3}-\frac{x}{4}+\frac{x-2}{5}=3.$

43. $\frac{7x+5}{6}-\frac{5x+6}{4}=\frac{8-5x}{12}.$

44. $\frac{x+4}{3}-\frac{x-4}{5}=2+\frac{3x-1}{15}.$

45. $\frac{x-1}{2}+\frac{2x+7}{3}-\frac{x+2}{9}=9.$

46. $\frac{x-1}{2}-\frac{x-2}{3}+\frac{x-3}{4}=\frac{2}{3}.$

47. $\frac{2x-5}{6}+\frac{6x+3}{4}=5x-17\frac{1}{2}.$

48. $\frac{x}{4}-\frac{5x+8}{6}=\frac{2x-9}{3}.$

49. $\frac{3x+5}{7}-\frac{2x+7}{3}+10-\frac{3x}{5}=0.$

50. $\frac{1}{7}(3x-4)+\frac{1}{3}(5x+3)=43-5x.$

51. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}+\frac{x}{5}=7\frac{5}{8}.$

52. $\frac{x}{2}-\frac{x-2}{3}=\frac{x+3}{4}-\frac{2}{3}.$

53. $\frac{5-3x}{4}+\frac{5x}{3}=\frac{3}{2}-\frac{3-5x}{3}.$

54. $\frac{1}{2}(27-2x)=\frac{9}{2}-\frac{1}{10}(7x-54).$

55. $5x-[8x-3\{16-6x-(4-5x)\}]=6.$

56. $\frac{1-2x}{3}-\frac{4-5x}{6}+\frac{13}{42}=0.$

57. $\frac{x+1}{3}-\frac{x-1}{4}+4x=12+\frac{2x-1}{6}.$

58. $\frac{4x-7}{8}+2\frac{2}{3}+\frac{7-4x}{4}=x+\frac{13}{24}.$

59. $\frac{5x-1}{7}+\frac{9x-5}{11}=\frac{9x-7}{5}.$

60. $\frac{x+3}{2}-\frac{x-2}{3}=\frac{3x-5}{12}+\frac{1}{4}.$

61. $\frac{1}{6}(8-x)+x-1\frac{2}{3}=\frac{1}{2}(x+6)-\frac{x}{3}.$

62. $\frac{3x-1}{5}-\frac{13-x}{2}=\frac{7x}{3}-\frac{11}{6}(x+3).$

63. $\frac{2x-1}{5}+\frac{6x-4}{7}=\frac{7x+12}{11}.$

64. $\frac{7x-4}{8}+2\frac{2}{3}+\frac{4-7x}{4}=x-\frac{7}{12}.$

65. $\frac{2-x}{3}+\frac{3-x}{4}+\frac{4-x}{5}+\frac{5-x}{6}+\frac{3}{4}=0.$

66. $\frac{5x-3}{7}-\frac{9-x}{3}=\frac{5x}{2}+\frac{19}{6}(x-4).$

XX. *Simple Equations, continued.*

170. We shall now give some examples of the solution of simple equations, which are a little more difficult than those in the preceding Chapter. The student will see that it is sometimes advantageous to clear of fractions *partially*, and then to effect some reductions, before we remove the remaining fractions.

171. Solve $\frac{x+6}{11} - \frac{2x-18}{3} + \frac{2x+3}{4} = 5\frac{1}{3} + \frac{3x+4}{12}$.

Here we may conveniently multiply by 12; thus,

$$\frac{12(x+6)}{11} - 4(2x-18) + 3(2x+3) = \frac{16}{3} \times 12 + 3x + 4,$$

that is, $\frac{12(x+6)}{11} - 8x + 72 + 6x + 9 = 64 + 3x + 4.$

By transposition and reduction we obtain

$$\frac{12(x+6)}{11} = 5x - 13.$$

Multiply by 11; thus $12(x+6) = 11(5x-13)$,

that is, $12x + 72 = 55x - 143;$

by transposition, $72 + 143 = 55x - 12x,$

that is, $43x = 215;$

therefore $x = \frac{215}{43} = 5.$

172. Solve $\frac{6x - 13\frac{1}{3}}{15 - 2x} + 2x + \frac{16x - 15}{24} = 6\frac{5}{12} - \frac{20\frac{5}{8} - 8x}{3}$.

Here we may conveniently multiply by 24; thus

$$\frac{24\left(6x - \frac{40}{3}\right)}{15 - 2x} + 48x + 16x - 15 = 24 \times \frac{77}{12} - 8\left(\frac{165}{8} - 8x\right);$$

that is,

$$\frac{144x-320}{15-2x}+48x+16x-15=154-165+64x.$$

By transposition and reduction

$$\frac{144x-320}{15-2x}=4;$$

multiply by $15-2x$; thus

$$144x-320=4(15-2x)=60-8x;$$

therefore $$144x+8x=320+60,$$

that is, $$152x=380;$$

therefore $$x=\frac{380}{152}=2\tfrac{76}{152}=2\tfrac{1}{2}.$$

173. Solve $\dfrac{x-5}{x-7}=\dfrac{x+3}{x+9}$.

Multiply by $(x-7)(x+9)$; thus

$$(x+9)(x-5)=(x-7)(x+3),$$

that is, $$x^2+4x-45=x^2-4x-21;$$

subtract x^2 from each side of the equation, thus

$$4x-45=-4x-21;$$

transpose, $$4x+4x=45-21,$$

that is, $$8x=24;$$

therefore $$x=\frac{24}{8}=3.$$

It will be seen that in this example x^2 is found on *both sides* of the equation, after we have cleared of fractions; accordingly it can be removed by subtraction, and so the equation remains a *simple equation.*

174. Solve $\frac{2x+3}{x+1}=\frac{4x+5}{4x+4}+\frac{3x+3}{3x+1}$.

Here it is convenient to multiply by $4x+4$, that is by $4(x+1)$;

thus $$4(2x+3)=4x+5+\frac{4(x+1)3(x+1)}{3x+1};$$

therefore $$8x+12-4x-5=\frac{12(x+1)^2}{3x+1};$$

that is, $$4x+7=\frac{12(x+1)^2}{3x+1}.$$

Multiply by $3x+1$; thus $(3x+1)(4x+7)=12(x+1)^2$;

that is, $$12x^2+25x+7=12x^2+24x+12.$$

Subtract $12x^2$ from each side, and transpose; thus

$$25x-24x=12-7,$$

that is, $$x=5.$$

175. Solve $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-4}{x-5}-\frac{x-5}{x-6}$.

We have $$\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{(x-1)(x-3)-(x-2)^2}{(x-2)(x-3)}$$

$$=\frac{x^2-4x+3-(x^2-4x+4)}{(x-2)(x-3)}=-\frac{1}{(x-2)(x-3)}.$$

And $$\frac{x-4}{x-5}-\frac{x-5}{x-6}=\frac{(x-4)(x-6)-(x-5)^2}{(x-5)(x-6)}$$

$$=\frac{x^2-10x+24-(x^2-10x+25)}{(x-5)(x-6)}=-\frac{1}{(x-5)(x-6)}.$$

Thus the proposed equation becomes

$$-\frac{1}{(x-2)(x-3)}=-\frac{1}{(x-5)(x-6)}.$$

Change the signs; thus $\frac{1}{(x-2)(x-3)} = \frac{1}{(x-5)(x-6)}$.

Clear of fractions; thus $(x-5)(x-6)=(x-2)(x-3)$;

that is, $x^2-11x+30=x^2-5x+6$;

therefore $-11x+5x=6-30$;

that is, $-6x=-24$;

therefore $6x=24$;

therefore $x=4$.

176. Solve $\cdot 5x+\frac{\cdot 45x-\cdot 75}{\cdot 6}=\frac{1\cdot 2}{\cdot 2}-\frac{\cdot 3x-\cdot 6}{\cdot 9}$.

To ensure accuracy it is advisable to express all the decimals as common fractions; thus

$$\frac{5x}{10}+\frac{10}{6}\left(\frac{45x}{100}-\frac{75}{100}\right)=\frac{10}{2}\times\frac{12}{10}-\frac{10}{9}\left(\frac{3x}{10}-\frac{6}{10}\right).$$

Simplifying, $\frac{x}{2}+\frac{5}{3}\left(\frac{9x}{20}-\frac{3}{4}\right)=6-\left(\frac{x}{3}-\frac{2}{3}\right)$;

that is, $\frac{x}{2}+\frac{3x}{4}-\frac{5}{4}=6-\frac{x}{3}+\frac{2}{3}$.

Multiply by 12, $6x+9x-15=72-4x+8$;

transpose, $19x=72+8+15=95$;

therefore $x=\frac{95}{19}=5$.

177. Equations may be proposed in which *letters* are used to represent known quantities; we shall continue to represent the unknown quantity by x, and any other letter will be supposed to represent a known quantity. We will solve three such equations.

178. Solve $\frac{x}{a} + \frac{x}{b} = c.$

Multiply by ab; thus $bx + ax = abc$;

that is, $$(a+b)x = abc;$$

divide by $a+b$; thus $$x = \frac{abc}{a+b}.$$

179. Solve $(a+x)(b+x) = a(b+c) + \frac{a^2c}{b} + x^2.$

Here $ab + ax + bx + x^2 = ab + ac + \frac{a^2c}{b} + x^2$;

therefore $$ax + bx = ac + \frac{a^2c}{b};$$

that is, $$(a+b)x = ac\left(1 + \frac{a}{b}\right) = \frac{ac(a+b)}{b};$$

divide by $a+b$; thus $$x = \frac{ac}{b}.$$

180. Solve $$\frac{x-a}{x-b} = \frac{(2x-a)^2}{(2x-b)^2}.$$

Clear of fractions; thus

$$(x-a)(2x-b)^2 = (x-b)(2x-a)^2;$$

that is, $(x-a)(4x^2 - 4xb + b^2) = (x-b)(4x^2 - 4xa + a^2)$.

Multiplying out we obtain

$$4x^3 - 4x^2(a+b) + x(4ab + b^2) - ab^2$$
$$= 4x^3 - 4x^2(a+b) + x(4ab + a^2) - a^2b;$$

therefore $$xb^2 - ab^2 = xa^2 - a^2b;$$

therefore $$x(a^2 - b^2) = a^2b - ab^2 = ab(a-b);$$

therefore $$x = \frac{ab(a-b)}{a^2 - b^2} = \frac{ab}{a+b}.$$

181. Although the following equation does not strictly belong to the present Chapter we give it as there will be no difficulty in following the steps of the solution, and it will serve as a model for similar examples. The equation resembles those already solved, in the circumstance that we obtain only a *single* value of the unknown quantity.

Solve $\sqrt{x} + \sqrt{(x-16)} = 8.$

By transposition, $\sqrt{(x-16)} = 8 - \sqrt{x}$;

square both sides; thus $x - 16 = (8 - \sqrt{x})^2 = 64 - 16\sqrt{x} + x$;

therefore $-16 = 64 - 16\sqrt{x}$;

transpose, $16\sqrt{x} = 64 + 16 = 80$;

therefore $\sqrt{x} = 5$;

therefore $x = 25.$

EXAMPLES. XX.

1. $\dfrac{12}{x} + \dfrac{1}{12x} = \dfrac{29}{24}.$

2. $\dfrac{42}{x-2} = \dfrac{35}{x-3}.$

3. $\dfrac{128}{3x-4} = \dfrac{216}{5x-6}.$

4. $\dfrac{45}{2x+3} = \dfrac{57}{4x-5}.$

5. $\dfrac{3x-1}{2} - \dfrac{2x-5}{3} + \dfrac{x-3}{4} - \dfrac{x}{6} = x+1.$

6. $\dfrac{\frac{1}{2}x-3}{5} + \dfrac{\frac{3}{4}x-10}{2} + \dfrac{4-x}{4} = \dfrac{10-x}{6}.$

7. $\dfrac{5}{6}\left(x-\dfrac{1}{3}\right) + \dfrac{7}{6}\left(\dfrac{x}{5}-\dfrac{1}{7}\right) = 4\tfrac{8}{9}.$

8. $x + \dfrac{5x-8}{3} = 6 - \dfrac{3x-8}{5}.$

9. $\dfrac{x-2}{4} + \dfrac{1}{3} = x - \dfrac{2x-1}{3}.$

10. $x+1-\dfrac{x^2+3}{x+2}=2.$

11. $\dfrac{x-1}{x-2}=\dfrac{7x-21}{7x-26}.$

12. $\dfrac{7x-4}{x-1}=\dfrac{7x-26}{x-3}.$

13. $\dfrac{x}{7}-\dfrac{3x}{2}+\dfrac{71}{7}=\dfrac{3x+1}{2}+1\frac{1}{14}.$

14. $\dfrac{2x-6}{3x-8}=\dfrac{2x-5}{3x-7}.$

15. $x-3-(3-x)(x+1)=x(x-3)+8.$

16. $3-x-2(x-1)(x+2)=(x-3)(5-2x).$

17. $\dfrac{7+9x}{4}-1+\dfrac{2-x}{9}=7x.$

18. $(x+7)(x+1)=(x+3)^2.$

19. $\dfrac{1}{3}(2x-10)-\dfrac{1}{11}(3x-40)=15-\dfrac{1}{5}(57-x).$

20. $\dfrac{6x+8}{2x+1}-\dfrac{2x+38}{x+12}=1.$

21. $\dfrac{x-1}{4}-\dfrac{x-5}{32}+\dfrac{15-2x}{40}=\dfrac{9-x}{2}-\dfrac{7}{8}.$

22. $\dfrac{4x+17}{x+3}+\dfrac{3x-10}{x-4}=7.$

23. $\dfrac{x+1}{7}+x(x-2)=(x-1)^2.$

24. $\dfrac{x-4}{3}+(x-1)(x-2)=x^2-2x-4.$

25. $\dfrac{3x^2-2x-8}{5}=\dfrac{(7x-2)(3x-6)}{35}.$

26. $\dfrac{x+10}{3}-\dfrac{2}{5}(3x-4)+\dfrac{(3x-2)(2x-3)}{6}=x^2-\dfrac{2}{15}.$

27. $\dfrac{3x-1}{2x-1}-\dfrac{4x-2}{3x-2}=\dfrac{1}{6}.$

28. $\dfrac{2}{2x-3}+\dfrac{1}{x-2}=\dfrac{6}{3x+2}.$

29. $\dfrac{x-4}{x-5}-\dfrac{x-5}{x-6}=\dfrac{x-7}{x-8}-\dfrac{x-8}{x-9}.$

30. $\dfrac{x}{x-2}+\dfrac{x-9}{x-7}=\dfrac{x+1}{x-1}+\dfrac{x-8}{x-6}.$

31. $\dfrac{3-2x}{1-2x}-\dfrac{2x-5}{2x-7}=1-\dfrac{4x^2-1}{7-16x+4x^2}.$

32. $\dfrac{3+x}{3-x}-\dfrac{2+x}{2-x}-\dfrac{1+x}{1-x}=1.$

33. $\dfrac{x-5}{7}+\dfrac{x^2+6}{3}=\dfrac{x^2-2}{2}-\dfrac{x^2-x+1}{6}+3.$

34. $(x+1)(x+2)(x+3)$
$=(x-1)(x-2)(x-3)+3(4x-2)(x+1).$

35. $(x-9)(x-7)(x-5)(x-1)$
$=(x-2)(x-4)(x-6)(x-10).$

36. $(8x-3)^2(x-1)=(4x-1)^2(4x-5).$

37. $\dfrac{x^2-x+1}{x-1}+\dfrac{x^2+x+1}{x+1}=2x.$

38. $\cdot 5x-2=\cdot 25x+\cdot 2x-1.$

39. $\cdot 5x+\cdot 6x-\cdot 8=\cdot 75x+\cdot 25.$

40. $\cdot 15x+\dfrac{\cdot 135x-\cdot 225}{\cdot 6}=\dfrac{\cdot 36}{\cdot 2}-\dfrac{\cdot 09x-\cdot 18}{\cdot 9}.$

41. $a\dfrac{a-x}{b}-b\dfrac{b+x}{a}=x.$

42. $a\dfrac{x-a}{b}+b\dfrac{x-b}{a}=x.$

43. $\dfrac{x^2-a^2}{bx}-\dfrac{a-x}{b}=\dfrac{2x}{b}-\dfrac{a}{x}.$

44. $x(x-a)+x(x-b)=2(x-a)(x-b).$

45. $(x-a)(x-b)(x+2a+2b)$
$=(x+2a)(x+2b)(x-a-b).$

46. $(x-a)(x-b)=(x-a-b)^2.$

47. $\dfrac{a}{x-a}-\dfrac{b}{x-b}=\dfrac{a-b}{x-c}.$

48. $\dfrac{a}{x+a}+\dfrac{b}{x+b}=\dfrac{a+b}{x+c}.$

49. $\dfrac{1}{x-a}-\dfrac{1}{x-b}=\dfrac{a-b}{x^2-ab}.$

50. $\dfrac{1}{x-a}-\dfrac{1}{x-a+c}=\dfrac{1}{x-b-c}-\dfrac{1}{x-b}.$

51. $\dfrac{mx-a-b}{nx-c-d}=\dfrac{mx-a-c}{nx-b-d}.$

52. $(a-b)(x-c)-(b-c)(x-a)-(c-a)(x-b)=0.$

53. $\dfrac{x-a}{a-b}-\dfrac{x+a}{a+b}=\dfrac{2ax}{a^2-b^2}.$

54. $(a-x)(b-x)=(p+x)(q+x).$

55. $\dfrac{x-a}{x-a-1}-\dfrac{x-a-1}{x-a-2}=\dfrac{x-b}{x-b-1}-\dfrac{x-b-1}{x-b-2}.$

56. $(x+a)(2x+b+c)^2=(x+b)(2x+a+c)^2.$

57. $(x+2a)(x-a)^2=(x+2b)(x-b)^2.$

58. $(x-a)^3(x+a-2b)=(x-b)^3(x-2a+b).$

59. $\sqrt{(4x)}+\sqrt{(4x-7)}=7.$

60. $\sqrt{(x+14)}+\sqrt{(x-14)}=14.$

61. $\sqrt{(x+11)}+\sqrt{(x-9)}=10.$

62. $\sqrt{(9x+4)}+\sqrt{(9x-1)}=3.$

63. $\sqrt{(x+4ab)}=2a-\sqrt{x}.$

64. $\sqrt{(x-a)}+\sqrt{(x-b)}=\sqrt{(a-b)}.$

XXI. *Problems.*

182. We shall now apply the methods explained in the preceding two Chapters to the solution of some problems, and thus exhibit to the student specimens of the use of Algebra. In these problems certain quantities are given, and another, which has some assigned relations to these, has to be found; the quantity which has to be found is called the *unknown quantity.* The relations are usually expressed in ordinary language in the enunciation of the problem, and the method of solving the problem may be thus described in general terms: *denote the unknown quantity by the letter* x, *and express in algebraical language the relations which hold between the unknown quantity and the given quantities; an equation will thus be obtained from which the value of the unknown quantity may be found.*

183. The sum of two numbers is 85, and their difference is 27: find the numbers.

Let x denote the less number; then, since the difference of the numbers is 27, the greater number will be denoted by $x+27$; and since the sum of the numbers is 85 we have

$$x+x+27=85;$$

that is, $$2x+27=85;$$

therefore $$2x=85-27=58;$$

therefore $$x=\frac{58}{2}=29.$$

Thus the less number is 29; and the greater number is $29+27$, that is 56.

184. Divide £2. 10*s*. among A, B, and C, so that B may have 5*s*. more than A, and C may have as much as A and B together.

Let x denote the number of shillings in A's share, then $x+5$ will denote the number of shillings in B's share, and $2x+5$ will denote the number of shillings in C's share.

The whole number of shillings is 50; therefore

$$x + x + 5 + 2x + 5 = 50;$$

that is, $$4x + 10 = 50;$$

therefore $$4x = 50 - 10 = 40;$$

therefore $$x = 10.$$

Thus A's share is 10 shillings, B's share is 15 shillings, and C's share is 25 shillings.

185. A certain sum of money was divided between A, B, and C; A and B together received £17. 15*s.*; A and C together received £15. 15*s.*; B and C together received £12. 10*s.*: find the sum received by each.

Let x denote the number of pounds which A received, then B received $17\frac{3}{4} - x$ pounds, because A and B together received $17\frac{3}{4}$ pounds; and C received $15\frac{3}{4} - x$ pounds, because A and C together received $15\frac{3}{4}$ pounds. Also B and C together received $12\frac{1}{2}$ pounds; therefore

$$12\tfrac{1}{2} = 17\tfrac{3}{4} - x + 15\tfrac{3}{4} - x;$$

that is, $$12\tfrac{1}{2} = 33\tfrac{1}{2} - 2x;$$

therefore $$2x = 33\tfrac{1}{2} - 12\tfrac{1}{2} = 21:$$

therefore $$x = \frac{21}{2} = 10\tfrac{1}{2}.$$

Thus A received £10. 10*s.*, B received £7. 5*s.*, and C received £5. 5*s.*

186. A grocer has some tea worth 2*s.* a lb., and some worth 3*s.* 6*d.* a lb.: how many lbs. must he take of each sort to produce 100 lbs. of a mixture worth 2*s.* 6*d.* a lb.?

Let x denote the number of lbs. of the first sort; then $100 - x$ will denote the number of lbs. of the second sort. The value of the x lbs. is $2x$ shillings; and the value of the

$100-x$ lbs. is $\frac{7}{2}(100-x)$ shillings. And the whole value is to be $\frac{5}{2}\times 100$ shillings; therefore

$$\frac{5}{2}\times 100 = 2x + \frac{7}{2}(100-x);$$

multiply by 2, thus $500 = 4x + 700 - 7x$;

therefore $7x - 4x = 700 - 500$;

that is, $3x = 200$;

therefore $x = \frac{200}{3} = 66\frac{2}{3}$.

Thus there must be $66\frac{2}{3}$ lbs. of the first sort, and $33\frac{1}{3}$ lbs. of the second sort.

187. A line is 2 feet 4 inches long; it is required to divide it into two parts, such that one part may be three-fourths of the other part.

Let x denote the number of inches in the larger part; then $\frac{3x}{4}$ will denote the number of inches in the other part.

The number of inches in the whole line is 28; therefore

$$x + \frac{3x}{4} = 28;$$

therefore $4x + 3x = 112$;

that is, $7x = 112$;

therefore $x = 16$.

Thus one part is 16 inches long, and the other part 12 inches long.

188. A person had £1000, part of which he lent at 4 per cent., and the rest at 5 per cent.; the whole annual interest received was £44: how much was lent at 4 per cent.?

Let x denote the number of pounds lent at 4 per cent.; then $1000-x$ will denote the number of pounds lent at 5 per cent. The annual interest obtained from the former is $\frac{4x}{100}$, and from the latter $\frac{5(1000-x)}{100}$;

therefore $$44=\frac{4x}{100}+\frac{5(1000-x)}{100};$$

therefore $$4400=4x+5(1000-x);$$

that is, $$4400=4x+5000-5x;$$

therefore $$x=5000-4400=600.$$

Thus £600 was lent at 4 per cent.

189. The student will find that the only difficulty in solving a problem consists in translating statements expressed in ordinary language into Algebraical language; and he should not be discouraged, if he is sometimes a little perplexed, since nothing but practice can give him readiness and certainty in this process. One remark may be made, which is very important for beginners; what is called the unknown *quantity* is really an unknown *number*, and this should be distinctly noticed in forming the equation. Thus, for example, in the second problem which we have solved, we begin by saying, let x denote the number of shillings in A's share; beginners often say, let $x=A$'s money, which is not definite, because A's money may be expressed in various ways, in pounds, or in shillings, or as a fraction of the whole sum. Again, in the fifth problem which we have solved, we begin by saying, let x denote the number of inches in the longer part; beginners often say, let $x=$ the longer part, or, let $x=$ a part, and to these phrases the same objection applies as to that already noticed.

190. Beginners often find a difficulty in translating a problem from ordinary language into Algebraical language, because they do not understand what is meant by the ordinary language. If no consistent meaning can be assigned to the words, it is of course impossible to translate them; but it often happens that the words are not ab-

solutely unintelligible, but appear to be susceptible of more than one meaning. The student should then select one meaning, express that meaning in Algebraical symbols, and deduce from it the result to which it will lead. If the result be inadmissible, or absurd, the student should try another meaning of the words. But if the result is satisfactory he may infer that he has probably understood the words correctly; though it may still be interesting to try the other possible meanings, in order to see if the enunciation really is susceptible of more than one meaning.

191. A student in solving the problems which are given for exercise, may find some which he can readily solve by Arithmetic, or by a process of guess and trial; and he may be thus inclined to undervalue the power of Algebra, and look on its aid as unnecessary. But we may remark that by Algebra the student is enabled to solve *all* these problems, without any uncertainty; and moreover, he will find as he proceeds, that by Algebra he can solve problems which would be extremely difficult or altogether impracticable, if he relied on Arithmetic alone.

Examples. XXI.

1. Find the number which exceeds its fifth part by 24.

2. A father is 30 years old, and his son is 2 years old: in how many years will the father be eight times as old as the son?

3. The difference of two numbers is 7, and their sum is 33: find the numbers.

4. The sum of £155 was raised by A, B, and C together; B contributed £15 more than A, and C £20 more than B: how much did each contribute?

5. The difference of two numbers is 14, and their sum is 48: find the numbers.

6. A is twice as old as B, and seven years ago their united ages amounted to as many years as now represent the age of A: find the ages of A and B.

7. If 56 be added to a certain number, the result is treble that number: find the number.

8. A child is born in November, and on the tenth day of December he is as many days old as the month was on the day of his birth: when was he born?

9. Find that number the double of which increased by 24 exceeds 80 as much as the number itself is below 100.

10. There is a certain fish, the head of which is 9 inches long; the tail is as long as the head and half the back; and the back is as long as the head and tail together: what is the length of the back and of the tail?

11. Divide the number 84 into two parts such that three times one part may be equal to four times the other.

12. The sum of £76 was raised by A, B, and C together; B contributed as much as A and £10 more, and C as much as A and B together: how much did each contribute?

13. Divide the number 60 into two parts such that a seventh of one part may be equal to an eighth of the other part.

14. After 34 gallons had been drawn out of one of two equal casks, and 80 gallons out of the other, there remained just three times as much in one cask as in the other: what did each cask contain when full?

15. Divide the number 75 into two parts such that 3 times the greater may exceed 7 times the less by 15.

16. A person distributes 20 shillings among 20 persons, giving sixpence each to some, and sixteen pence each to the rest: how many persons received sixpence each?

17. Divide the number 20 into two parts such that the sum of three times one part, and five times the other part, may be 84.

18. The price of a work which comes out in parts is £2. 16*s.* 8*d.*; but if the price of each part were 13 pence more than it is, the price of the work would be £3. 7*s.* 6*d.*: how many parts were there?

19. Divide 45 into two parts such that the first divided by 2 shall be equal to the second multiplied by 2.

20. A father is three times as old as his son; four years ago the father was four times as old as his son then was: what is the age of each?

21. Divide 188 into two parts such that the fourth of one part may exceed the eighth of the other by 14.

22. A person meeting a company of beggars gave four pence to each, and had sixteen pence left; he found that he should have required a shilling more to enable him to give the beggars sixpence each: how many beggars were there?

23. Divide 100 into two parts such that if a third of one part be subtracted from a fourth of the other the remainder may be 11.

24. Two persons, A and B, engage at play; A has £72 and B has £52 when they begin, and after a certain number of games have been won and lost between them, A has three times as much money as B: how much did A win?

25. Divide 60 into two parts such that the difference between the greater and 64 may be equal to twice the difference between the less and 38.

26. The sum of £276 was raised by A, B, and C together; B contributed twice as much as A and £12 more; and C three times as much as B and £12 more: how much did each contribute?

27. Find a number such that the sum of its fifth and its seventh shall exceed the sum of its eighth and its twelfth by 113.

28. An army in a defeat loses one-sixth of its number in killed and wounded, and 4000 prisoners; it is reinforced by 3000 men, but retreats, losing one-fourth of its number in doing so; there remain 18000 men: what was the original force?

29. Find a number such that the sum of its fifth and its seventh shall exceed the difference of its fourth and its seventh by 99.

30. One-half of a certain number of persons received eighteen-pence each, one-third received two shillings each, and the rest received half a crown each; the whole sum distributed was £2. 4*s.*: how many persons were there?

31. A person had £900; part of it he lent at the rate of 4 per cent., and part at the rate of 5 per cent., and he received equal sums as interest from the two parts: how much did he lend at 4 per cent.?

32. A father has six sons, each of whom is four years older than his next younger brother; and the eldest is three times as old as the youngest: find their respective ages.

33. Divide the number 92 into four such parts that the first may exceed the second by 10, the third by 18, and the fourth by 24.

34. A gentleman left £550 to be divided among four servants *A*, *B*, *C*, *D*; of whom *B* was to have twice as much as *A*, *C* as much as *A* and *B* together, and *D* as much as *C* and *B* together: how much had each?

35. Find two consecutive numbers such that the half and the fifth of the first taken together shall be equal to the third and the fourth of the second taken together.

36. A sum of money is to be distributed among three persons *A*, *B*, and *C*; the shares of *A* and *B* together amount to £60; those of *A* and *C* to £80; and those of *B* and *C* to £92: find the share of each person.

37. Two persons *A* and *B* are travelling together; *A* has £100, and *B* has £48; they are met by robbers who take twice as much from *A* as from *B*, and leave to *A* three times as much as to *B*: how much was taken from each?

38. The sum of £500 was divided among four persons, so that the first and second together received £280, the first and third together £260, and the first and fourth together £220: find the share of each.

39. After *A* has received £10 from *B* he has as much money as *B* and £6 more; and between them they have £40: what money had each at first?

40. A wine merchant has two sorts of wines, one sort worth 2 shillings a quart, and the other worth 3*s*. 4*d*. a quart; from these he wants to make a mixture of 100 quarts worth 2*s*. 4*d*. a quart: how many quarts must he take from each sort?

41. In a mixture of wine and water the wine composed 25 gallons more than half of the mixture, and the water 5 gallons less than a third of the mixture: how many gallons were there of each?

42. In a lottery consisting of 10000 tickets, half the number of prizes added to one-third the number of blanks was 3500: how many prizes were there in the lottery?

43. In a certain weight of gunpowder the saltpetre composed 6 lbs. more than a half of the weight, the sulphur 5 lbs. less than a third, and the charcoal 3 lbs. less than a fourth: how many lbs. were there of each of the three ingredients?

44. A general, after having lost a battle, found that he had left fit for action 3600 men more than half of his army; 600 men more than one-eighth of his army were wounded; and the remainder, forming one-fifth of the army, were slain, taken prisoners, or missing: what was the number of the army?

45. How many sheep must a person buy at £7 each that after paying one shilling a score for folding them at night he may gain £79. 16*s.* by selling them at £8 each?

46. A certain sum of money was shared among five persons *A*, *B*, *C*, *D*, and *E*; *B* received £10 less than *A*; *C* received £16 more than *B*; *D* received £5 less than *C*; and *E* received £15 more than *D*; and it was found that *E* received as much as *A* and *B* together: how much did each receive?

47. A tradesman starts with a certain sum of money; at the end of the first year he had doubled his original stock, all but £100; also at the end of the second year he had doubled the stock at the beginning of the second year, all but £100; also in like manner at the end of the third year; and at the end of the third year he was three times as rich as at first: find his original stock.

48. A person went to a tavern with a certain sum of money; there he borrowed as much as he had about him, and spent a shilling out of the whole; with the remainder he went to a second tavern, where he borrowed as much as he had left, and also spent a shilling; and he then went to a third tavern, borrowing and spending as before, after which he had nothing left: how much had he at first?

XXII. *Problems, continued.*

192. We shall now give some examples in which the process of translation from ordinary language to algebraical language is rather more difficult than in the examples of the preceding Chapter.

193. It is required to divide the number 80 into four such parts, that the first increased by 3, the second diminished by 3, the third multiplied by 3, and the fourth divided by 3 may all be equal.

Let the number x denote the first part; then if it be increased by 3 we obtain $x+3$, and this is to be equal to the second part diminished by 3, so that the second part must be $x+6$; again, $x+3$ is to be equal to the third part multiplied by 3, so that the third part must be $\frac{x+3}{3}$; and $x+3$ is to be equal to the fourth part divided by 3, so that the fourth part must be $3(x+3)$. And the sum of the parts is to be equal to 80.

Therefore $$x+x+6+\frac{x+3}{3}+3(x+3)=80,$$

that is, $$2x+6+\frac{x+3}{3}+3x+9=80,$$

that is, $$5x+\frac{x+3}{3}=80-15=65;$$

multiply by 3; thus $$15x+x+3=195,$$

that is, $$16x=192;$$

therefore $$x=\frac{192}{16}=12.$$

Thus the parts are 12, 18, 5, 45.

194. *A* alone can perform a piece of work in 9 days, and *B* alone can perform it in 12 days: in what time will they perform it if they work together?

Let x denote the required number of days. In one day *A* can perform $\frac{1}{9}$th of the work; therefore in x days he can perform $\frac{x}{9}$ths of the work. In one day *B* can perform $\frac{1}{12}$th of the work; therefore in x days he can perform $\frac{x}{12}$ths of the work. And since in x days *A* and *B* together perform the *whole* work, the sum of the *fractions* of the work must be equal to *unity;* that is,

$$\frac{x}{9} + \frac{x}{12} = 1.$$

Multiply by 36; thus $$4x + 3x = 36,$$

that is, $$7x = 36;$$

therefore $$x = \frac{36}{7} = 5\tfrac{1}{7}.$$

195. A cistern could be filled with water by means of one pipe alone in 6 hours, and by means of another pipe alone in 8 hours; and it could be emptied by a tap in 12 hours if the two pipes were closed: in what time will the cistern be filled if the pipes and the tap are all open?

Let x denote the required number of hours. In one hour the first pipe fills $\frac{1}{6}$th of the cistern; therefore in x hours it fills $\frac{x}{6}$ths of the cistern. In one hour the second pipe fills $\frac{1}{8}$th of the cistern; therefore in x hours it fills $\frac{x}{8}$ths of the cistern. In one hour the tap empties $\frac{1}{12}$th

of the cistern; therefore in x hours it empties $\frac{x}{12}$ths of the cistern. And since in x hours the *whole* cistern is filled, we have

$$\frac{x}{6}+\frac{x}{8}-\frac{x}{12}=1.$$

Multiply by 24; thus $4x+3x-2x=24$,

that is, $5x=24$;

therefore $x=\frac{24}{5}=4\frac{4}{5}$.

196. It is sometimes convenient to denote by x, not the unknown quantity which is explicitly required, but some other quantity from which that can be easily deduced; this will be illustrated in the next two problems.

197. A colonel on attempting to draw up his regiment in the form of a solid square finds that he has 31 men over, and that he would require 24 men more in his regiment in order to increase the side of the square by one man: how many men were there in the regiment?

Let x denote the number of men in the side of the first square; then the number of men in the square is x^2 and the number of men in the regiment is x^2+31. If there were $x+1$ men in a side of the square, the number of men in the square would be $(x+1)^2$; thus the number of men in the regiment is $(x+1)^2-24$.

Therefore $(x+1)^2-24=x^2+31$,

that is, $x^2+2x+1-24=x^2+31$.

From these two equal expressions we can remove x^2 which occurs in both; thus

$$2x+1-24=31;$$

therefore $2x=31-1+24=54$;

therefore $x=\frac{54}{2}=27$.

Hence the number of men in the regiment is $(27)^2+31$, that is, $729+31$, that is, 760.

198. A starts from a certain place, and travels at the rate of 7 miles in 5 hours; B starts from the same place 8 hours after A, and travels in the same direction at the rate of 5 miles in 3 hours: how far will A travel before he is overtaken by B?

Let x represent the number of hours which A travels before he is overtaken; therefore B travels $x-8$ hours. Now since A travels 7 miles in 5 hours, he travels $\frac{7}{5}$ of a mile in one hour; and therefore in x hours he travels $\frac{7x}{5}$ miles. Similarly B travels $\frac{5}{3}$ of a mile in one hour, and therefore in $x-8$ hours he travels $\frac{5}{3}(x-8)$ miles. And when B overtakes A they have travelled the same number of miles. Therefore

$$\frac{5}{3}(x-8)=\frac{7x}{5};$$

multiply by 15; thus $25(x-8)=21x$,

that is, $25x-200=21x$;

therefore $25x-21x=200$,

that is, $4x=200$;

therefore $x=\frac{200}{4}=50$.

Therefore $\frac{7x}{5}=\frac{7}{5}\times 50=70$; so that A travelled 70 miles before he was overtaken.

199. Problems are sometimes given which suppose the student to have obtained from Arithmetic a knowledge of

the meaning of *proportion;* this will be illustrated in the next two problems. After them we shall conclude the Chapter with three problems of a more difficult character than those hitherto given.

200. It is required to divide the number 56 into two parts such that one may be to the other as 3 to 4.

Let the number x denote the first part; then the other part must be $56 - x$; and since x is to be to $56 - x$ as 3 to 4 we have

$$\frac{x}{56-x} = \frac{3}{4}.$$

Clear of fractions; thus

$$4x = 3(56 - x);$$

that is, $$4x = 168 - 3x;$$

therefore $$7x = 168;$$

therefore $$x = \frac{168}{7} = 24.$$

Thus the first part is 24 and the other part is $56 - 24$, that is 32.

The preceding method of solution is the most natural for a beginner; the following however is much shorter.

Let the number $3x$ denote the first part; then the second part must be $4x$, because the first part is to the second as 3 to 4. Then the sum of the two parts is equal to 56; thus

$$3x + 4x = 56,$$

that is, $$7x = 56;$$

therefore $$x = 8.$$

Thus the first part is 3×8, that is 24; and the second part is 4×8, that is 32.

201. A cask, A, contains 12 gallons of wine and 18 gallons of water; and another cask, B, contains 9 gallons of wine and 3 gallons of water: how many gallons must be drawn from each cask so as to produce by their mixture 7 gallons of wine and 7 gallons of water?

Let x denote the number of gallons to be drawn from A; then since the mixture is to consist of 14 gallons, $14-x$ will denote the number of gallons to be drawn from B. Now the number of gallons in A is 30, of which 12 are wine; that is, the wine is $\frac{12}{30}$ of the whole. Therefore the x gallons drawn from A contain $\frac{12x}{30}$ gallons of wine. Similarly the $14-x$ gallons drawn from B contain $\frac{9(14-x)}{12}$ gallons of wine. And the mixture is to contain 7 gallons of wine; therefore

$$\frac{12x}{30}+\frac{9(14-x)}{12}=7;$$

that is,
$$\frac{2x}{5}+\frac{3(14-x)}{4}=7;$$

therefore
$$8x+15(14-x)=140,$$

that is,
$$8x+210-15x=140;$$

therefore
$$7x=70;$$

therefore
$$x=10.$$

Thus 10 gallons must be drawn from A, and 4 from B.

202. At what time between 2 o'clock and 3 o'clock is one hand of a watch exactly over the other?

Let x denote the required number of minutes after 2 o'clock. In x minutes the long hand will move over x divisions of the watch face; and as the long hand moves twelve times as fast as the short hand, the short hand will move over $\frac{x}{12}$ divisions in x minutes. At 2 o'clock the

short hand is 10 divisions in advance of the long hand; so that in the x minutes the long hand must pass over 10 more divisions than the short hand; therefore

$$x = \frac{x}{12} + 10;$$

therefore $$12x = x + 120;$$

therefore $$11x = 120;$$

therefore $$x = \frac{120}{11} = 10\tfrac{10}{11}.$$

203. A hare takes four leaps to a greyhound's three, but two of the greyhound's leaps are equivalent to three of the hare's; the hare has a start of fifty leaps: how many leaps must the greyhound take to catch the hare?

Suppose that $3x$ denote the number of leaps taken by the greyhound; then $4x$ will denote the number of leaps taken by the hare in the same time. Let a denote the number of inches in one leap of the hare; then $3a$ denotes the number of inches in three leaps of the hare, and therefore also the number of inches in two leaps of the greyhound; therefore $\frac{3a}{2}$ denotes the number of inches in one leap of the greyhound. Then $3x$ leaps of the greyhound will contain $3x \times \frac{3a}{2}$ inches. And $50 + 4x$ leaps of the hare will contain $(50 + 4x)a$ inches; therefore

$$\frac{9xa}{2} = (50 + 4x)a.$$

Divide by a; thus $\frac{9x}{2} = 50 + 4x;$

therefore $$9x = 100 + 8x;$$

therefore $$x = 100.$$

Thus the greyhound must take 300 leaps.

The student will see that we have introduced an auxiliary symbol a, to enable us to form the equation easily; and that we can remove it by division when the equation is formed.

204. Four gamesters, A, B, C, D, each with a different stock of money, sit down to play; A wins half of B's first stock, B wins a third part of C's, C wins a fourth part of D's, and D wins a fifth part of A's; and then each of the gamesters has £23. Find the stock of each at first.

Let x denote the number of pounds which D won from A; then $5x$ will denote the number in A's first stock. Thus $4x$, together with what A won from B, make up 23; therefore $23-4x$ denotes the number of pounds which A won from B. And, since A won half of B's stock, $23-4x$ also denotes what was left with B after his loss to A.

Again, $23-4x$, together with what B won from C, make up 23; therefore $4x$ denotes the number of pounds which B won from C. And, since B won a third of C's first stock, $12x$ denotes C's first stock; and therefore $8x$ denotes what was left with C after his loss to B.

Again, $8x$, together with what C won from D, make up 23; therefore $23-8x$ denotes the number of pounds which C won from D. And, since C won a fourth of D's first stock, $4(23-8x)$ denotes D's first stock; and therefore $3(23-8x)$ denotes what was left with D after his loss to C.

Finally, $3(23-8x)$, together with x, which D won from A, make up 23; thus

$$23=3(23-8x)+x;$$

therefore $$23x=46;$$

therefore $$x=2.$$

Thus the stocks at first were 10, 30, 24, 28.

EXAMPLES. XXII.

1. A privateer running at the rate of 10 miles an hour discovers a ship 18 miles off, running at the rate of 8 miles an hour: how many miles can the ship run before it is overtaken?

2. Divide the number 50 into two parts such that if three-fourths of one part be added to five-sixths of the other part the sum may be 40.

3. Suppose the distance between London and Edinburgh is 360 miles, and that one traveller starts from Edinburgh and travels at the rate of 10 miles an hour, while another starts at the same time from London and travels at the rate of 8 miles an hour: it is required to know where they will meet.

4. Find two numbers whose difference is 4, and the difference of their squares 112.

5. A sum of 24 shillings is received from 24 people; some contribute 9*d.* each, and some $13\frac{1}{2}$*d.* each: how many contributors were there of each kind?

6. Divide the number 48 into two parts such that the excess of one part over 20 may be three times the excess of 20 over the other part.

7. A person has £98; part of it he lent at the rate of 5 per cent. simple interest, and the rest at the rate of 6 per cent. simple interest; and the interest of the whole in 15 years amounted to £81: how much was lent at 5 per cent.?

8. A person lent a certain sum of money at 6 per cent. simple interest; in 10 years the interest amounted to £12 less than the sum lent: what was the sum lent?

9. A person rents 25 acres of land for £7. 12*s.*; the land consists of two sorts, the better sort he rents at 8*s.* per acre, and the worse at 5*s.* per acre: how many acres are there of each sort?

10. A cistern could be filled in 12 minutes by two pipes which run into it; and it would be filled in 20 minutes by one alone: in what time could it be filled by the other alone?

11. Divide the number 90 into four parts such that the first increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2 may all be equal.

12. A person bought 30 lbs. of sugar of two different sorts, and paid for the whole 19*s.*; the better sort cost 10*d.* per lb., and the worse 7*d.* per lb.: how many lbs. were there of each sort?

13. Divide the number 88 into four parts such that the first increased by 2, the second diminished by 3, the third multiplied by 4, and the fourth divided by 5, may all be equal.

14. If 20 men, 40 women, and 50 children receive £50 among them for a week's work, and 2 men receive as much as 3 women or 5 children, what does each woman receive for a week's work?

15. Divide 100 into two parts such that the difference of their squares may be 1000.

16. There are two places 154 miles apart, from which two persons start at the same time with a design to meet; one travels at the rate of 3 miles in two hours, and the other at the rate of 5 miles in four hours: when will they meet?

17. Divide 44 into two parts such that the greater increased by 5 may be to the less increased by 7, as 4 is to 3.

18. *A* can do half as much work as *B*, *B* can do half as much as *C*, and together they can complete a piece of work in 24 days: in what time could each alone complete the work?

19. Divide the number 90 into four parts such that if the first be increased by 5, the second diminished by 4, the third multiplied by 3, and the fourth divided by 2, the results shall all be equal.

20. Three persons can together complete a piece of work in 60 days; and it is found that the first does three-fourths of what the second does, and the second four-fifths of what the third does: in what time could each one alone complete the work?

21. Divide the number 36 into two parts such that one part may be five-sevenths of the other.

22. A general on attempting to draw up his army in the form of a solid square finds that he has 60 men over, and that he would require 41 men more in his army in order to increase the side of the square by one man: how many men were there in the army?

23. Divide the number 90 into two parts such that one part may be two-thirds of the other.

24. A person bought a certain number of eggs, half of them at 2 a penny, and half of them at 3 a penny; he sold them again at the rate of 5 for two pence, and lost a penny by the bargain: what was the number of eggs?

25. A and B are at present of the same age; if A's age be increased by 36 years, and B's by 52 years, their ages will be as 3 to 4: what is the present age of each?

26. For 1 lb. of tea and 9 lbs. of sugar the charge is 8*s*. 6*d*.; for 1 lb. of tea and 15 lbs. of sugar the charge is 12*s*. 6*d*.: what is the price of 1 lb. of sugar?

27. A prize of £2000 was divided between A and B, so that their shares were in the proportion of 7 to 9: what was the share of each?

28. A workman was hired for 40 days at 3*s*. 4*d*. per day, for every day he worked; but with this condition that for every day he did not work he was to forfeit 1*s*. 4*d*.; and on the whole he had £3. 3*s*. 4*d*. to receive: how many days out of the 40 did he work?

29. A at play first won £5 from B, and had then as much money as B; but B, on winning back his own money and £5 more, had five times as much money as A: what money had each at first?

30. Divide 100 into two parts, such that the square of their difference may exceed the square of twice the less part by 2000.

31. A cistern has two supply pipes, which will singly fill it in $4\frac{1}{2}$ hours and 6 hours respectively; and it has also a leak by which it would be emptied in 5 hours: in how many hours will it be filled when all are working together?

32. A farmer would mix wheat at 4*s*. a bushel with rye at 2*s*. 6*d*. a bushel, so that the whole mixture may consist of 90 bushels, and be worth 3*s*. 2*d*. a bushel: how many bushels must be taken of each?

33. A bill of £3. 1*s*. 6*d*. was paid in half-crowns, and florins, and the whole number of coins was 28: how many coins were there of each kind?

34. A grocer with 56 lbs. of fine tea at 5*s*. a lb. would mix a coarser sort at 3*s*. 6*d*. a lb., so as to sell the whole together at 4*s*. 6*d*. a lb.: what quantity of the latter sort must he take?

35. A person hired a labourer to do a certain work on the agreement that for every day he worked he should receive 2*s*., but that for every day he was absent he should lose 9*d*.; he worked twice as many days as he was absent, and on the whole received £1. 19*s*.: find how many days he worked.

36. A regiment was drawn up in a solid square; when some time after it was again drawn up in a solid square it was found that there were 5 men fewer in a side; in the interval 295 men had been removed from the field: what was the original number of men in the regiment?

37. A sum of money was divided between *A* and *B*, so that the share of *A* was to that of *B* as 5 to 3; also the share of *A* exceeded five-ninths of the whole sum by £50: what was the share of each person?

38. A gentleman left his whole estate among his four sons. The share of the eldest was £800 less than half of the estate; the share of the second was £120 more than one-fourth of the estate; the third had half as much as the eldest; and the youngest had two-thirds of what the second had. How much did each son receive?

39. *A* and *B* began to play together with equal sums of money; *A* first won £20, but afterwards lost half of all he then had, and then his money was half as much as that of *B*: what money had each at first?

40. A lady gave a guinea in charity among a number of poor, consisting of men, women, and children; each man had 12*d*., each woman 6*d*., and each child 3*d*. The number of women was two less than twice the number of men; and the number of children four less than three times the number of women. How many persons were there relieved?

41. A draper bought a piece of cloth at 3*s.* 2*d.* per yard. He sold one-third of it at 4*s.* per yard, one-fourth of it at 3*s.* 8*d.* per yard, and the remainder at 3*s.* 4*d.* per yard; and his gain on the whole was 14*s.* 2*d.* How many yards did the piece contain?

42. A grazier spent £33. 7*s.* 6*d.* in buying sheep of different sorts. For the first sort, which formed one-third of the whole, he paid 9*s.* 6*d.* each. For the second sort, which formed one-fourth of the whole, he paid 11*s.* each. For the rest he paid 12*s.* 6*d.* each. What number of sheep did he buy?

43. A market woman bought a certain number of eggs, at the rate of 5 for twopence; she sold half of them at 2 a penny, and half of them at 3 a penny, and gained 4*d.* by so doing: what was the number of eggs?

44. A pudding consists of 2 parts of flour, 3 parts of raisins, and 4 parts of suet; flour costs 3*d.* a lb., raisins, 6*d.*, and suet 8*d.* Find the cost of the several ingredients of the pudding, when the whole cost is 2*s.* 4*d.*

45. Two persons, *A* and *B*, were employed together for 50 days, at 5*s.* per day each. During this time *A*, by spending 6*d.* per day less than *B*, saved twice as much as *B*, besides the expenses of two days over. How much did *A* spend per day?

46. Two persons, *A* and *B*, have the same income. *A* lays by one-fifth of his; but *B* by spending £60 per annum more than *A*, at the end of three years finds himself £100 in debt. What is the income of each?

47. *A* and *B* shoot by turns at a target. *A* puts 7 bullets out of 12 into the bull's eye, and *B* puts in 9 out of 12; between them they put in 32 bullets. How many shots did each fire?

48. Two casks, *A* and *B*, contain mixtures of wine and water; in *A* the quantity of wine is to the quantity of water as 4 to 3; in *B* the like proportion is that of 2 to 3. If *A* contain 84 gallons, what must *B* contain, so that when the two are put together, the new mixture may be half wine and half water?

49. The squire of a parish bequeaths a sum equal to one-hundredth part of his estate towards the restoration of the church; £200 less than this towards the endowment of the school; and £200 less than this latter sum towards the County Hospital. After deducting these legacies, $\frac{39}{40}$ of the estate remain to the heir. What was the value of the estate?

50. How many minutes does it want to 4 o'clock, if three-quarters of an hour ago it was twice as many minutes past two o'clock?

51. Two casks, A and B, are filled with two kinds of sherry, mixed in the cask A in the proportion of 2 to 7, and in the cask B in the proportion of 2 to 5: what quantity must be taken from each to form a mixture which shall consist of 2 gallons of the first kind and 6 of the second kind?

52. An officer can form the men of his regiment into a hollow square 12 deep. The number of men in the regiment is 1296. Find the number of men in the front of the hollow square.

53. A person buys a piece of land at £30 an acre, and by selling it in allotments finds the value increased threefold, so that he clears £150, and retains 25 acres for himself: how many acres were there?

54. The national debt of a country was increased by one-fourth in a time of war. During a long peace which followed £25000000 was paid off, and at the end of that time the rate of interest was reduced from $4\frac{1}{2}$ to 4 per cent. It was then found that the amount of annual interest was the same as before the war. What was the amount of the debt before the war?

55. A and B play at a game, agreeing that the loser shall always pay to the winner one shilling less than half the money the loser has; they commence with equal quantities of money, and after B has lost the first game and won the second, he has two shillings more than A: how much had each at the commencement?

56. A clock has two hands turning on the same centre; the swifter makes a revolution every twelve hours, and the slower every sixteen hours: in what time will the swifter gain just one complete revolution on the slower?

57. At what time between 3 o'clock and 4 o'clock is one hand of a watch exactly in the direction of the other hand produced?

58. The hands of a watch are at right angles to each other at 3 o'clock: when are they next at right angles?

59. A certain sum of money lent at simple interest amounted to £297. 12*s*. in eight months; and in seven more months it amounted to £306: what was the sum?

60. A watch gains as much as a clock loses; and 1799 hours by the clock are equivalent to 1801 hours by the watch: find how much the watch gains and the clock loses per hour.

61. It is between 11 and 12 o'clock, and it is observed that the number of minute spaces between the hands is two-thirds of what it was ten minutes previously: find the time.

62. A and B made a joint stock of £500 by which they gained £160, of which A had for his share £32 more than B: what did each contribute to the stock?

63. A distiller has 51 gallons of French brandy, which cost him 8 shillings a gallon; he wishes to buy some English brandy at 3 shillings a gallon to mix with the French, and sell the whole at 9 shillings a gallon. How many gallons of the English must he take, so that he may gain 30 per cent. on what he gave for the brandy of both kinds?

64. An officer can form his men into a hollow square 4 deep, and also into a hollow square 8 deep; the front in the latter formation contains 16 men fewer than in the former formation: find the number of men.

XXIII. *Simultaneous equations of the first degree with two unknown quantities.*

205. Suppose we have an equation containing two unknown quantities x and y, for example $3x-7y=8$. For every value which we please to assign to one of the unknown quantities we can determine the corresponding value of the other; and thus we can find as many pairs of values as we please which satisfy the given equation. Thus, for example, if $y=1$ we find $3x=15$, and therefore $x=5$; if $y=2$ we find $3x=22$, and therefore $x=7\frac{1}{3}$; and so on.

Also, suppose that there is another equation of the same kind, as for example $2x+5y=44$; then we can also find as many pairs of values as we please which satisfy this equation.

But suppose we ask for values of x and y which satisfy *both* equations; we shall find that there is only one value of x and one value of y. For multiply the first equation by 5; thus

$$15x-35y=40;$$

and multiply the second equation by 7; thus

$$14x+35y=308.$$

Therefore, by addition,

$$15x-35y+14x+35y=40+308;$$

that is,
$$29x=348;$$

therefore
$$x=\frac{348}{29}=12.$$

Thus if *both* equations are to be satisfied x *must* equal 12. Put this value of x in either of the two given equations, for example in the second; thus we obtain

$$24+5y=44;$$

therefore
$$5y=20;$$

therefore
$$y=4.$$

206. Two or more equations which are to be satisfied by the *same values* of the unknown quantities are called *simultaneous equations.* In the present Chapter we treat of simultaneous equations involving two unknown quantities, where each unknown quantity occurs only in the first degree, and the product of the unknown quantities does not occur.

207. There are three methods which are usually given for solving these equations. There is one principle common to all the methods; namely, from *two* given equations containing *two* unknown quantities a single equation is deduced containing only *one* of the unknown quantities. By this process we are said to *eliminate* the unknown quantity which does not appear in the single equation. The single equation containing only one unknown quantity can be solved by the method of Chapter XIX; and when the value of one of the unknown quantities has thus been determined, we can substitute this value in either of the given equations, and then determine the value of the other unknown quantity.

208. First method. *Multiply the equations by such numbers as will make the coefficient of one of the unknown quantities the same in the resulting equations; then by addition or subtraction we can form an equation containing only the other unknown quantity.*

This method we used in Art. 205; for another example, suppose

$$8x + 7y = 100,$$

$$12x - 5y = 88.$$

If we wish to eliminate y we multiply the first equation by 5, which is the coefficient of y in the second equation, and we multiply the second equation by 7, which is the coefficient of y in the first equation. Thus we obtain

$$40x + 35y = 500,$$

$$84x - 35y = 616;$$

therefore, by addition,

$$40x + 84x = 500 + 616;$$

that is, $$124x = 1116;$$

therefore $$x = 9.$$

Then put this value of x in either of the given equations, for example in the second; thus

$$108-5y=88;$$

therefore $$20=5y;$$

therefore $$y=4.$$

Suppose, however, that in solving these equations we wish to begin by eliminating x. If we multiply the first equation by 12, and the second by 8, we obtain

$$96x+84y=1200,$$

$$96x-40y=704.$$

Therefore, *by subtraction*,

$$84y+40y=1200-704;$$

that is, $$124y=496;$$

therefore $$y=4.$$

Or we may render the process more simple; for we may multiply the first equation by 3, and the second by 2; thus

$$24x+21y=300,$$

$$24x-10y=176.$$

Therefore, by subtraction,

$$21y+10y=300-176;$$

that is, $$31y=124;$$

therefore $$y=4.$$

209. Second method. *Express one of the unknown quantities in terms of the other from either equation, and substitute this value in the other equation.*

Thus, taking the example given in the preceding Article, we have from the first equation

$$8x=100-7y;$$

therefore $$x=\frac{100-7y}{8}.$$

Substitute this value of x in the second equation, and we obtain

$$\frac{12(100-7y)}{8}-5y=88;$$

that is, $$\frac{3(100-7y)}{2}-5y=88;$$

therefore $$3(100-7y)-10y=176;$$

that is, $$300-21y-10y=176;$$

therefore $$300-176=21y+10y;$$

that is, $$31y=124;$$

therefore $$y=4.$$

Then substitute this value of y in either of the given equations, and we shall obtain $x=9$.

Or thus: from the first equation we have

$$7y=100-8x;$$

therefore $$y=\frac{100-8x}{7}.$$

Substitute this value of y in the second equation, and we obtain

$$12x-\frac{5(100-8x)}{7}=88;$$

therefore $$84x-5(100-8x)=616;$$

that is, $$84x-500+40x=616;$$

therefore $$124x=500+616=1116;$$

therefore $$x=9.$$

210. Third method. *Express the same unknown quantity in terms of the other from each equation, and equate the expressions thus obtained.*

Thus, taking again the same example, from the first equation $x=\frac{100-7y}{8}$, and from the second equation $x=\frac{88+5y}{12}$.

Therefore $$\frac{100-7y}{8}=\frac{88+5y}{12}.$$

Clear of fractions, by multiplying by 24; thus

$$3(100-7y)=2(88+5y);$$

that is, $$300-21y=176+10y;$$

therefore $$300-176=21y+10y;$$

that is, $$31y=124;$$

therefore $$y=4.$$

Then, as before, we can deduce $x=9$.

Or thus: from the first equation $y=\frac{100-8x}{7}$, and from the second equation $y=\frac{12x-88}{5}$; therefore

$$\frac{100-8x}{7}=\frac{12x-88}{5}.$$

From this equation we shall obtain $x=9$; and then, as before, we can deduce $y=4$.

211. Solve $19x-21y=100$, $21x-19y=140$.

These equations may be solved by the methods already explained; we shall use them however to shew that these methods may be sometimes abbreviated.

Here, by addition, we obtain

$$19x-21y+21x-19y=100+140;$$

that is, $$40x-40y=240;$$

therefore $$x-y=6.$$

Again, from the original equations, by subtraction, we obtain

$$21x-19y-19x+21y=140-100;$$

that is, $$2x+2y=40;$$

therefore $$x+y=20.$$

Then since $x-y=6$ and $x+y=20$, we obtain by addition $2x=26$, and by subtraction $2y=14$;

therefore $$x=13, \text{ and } y=7.$$

212. The student will find as he proceeds that in all parts of Algebra, particular examples may be treated by methods which are shorter than the general rules; but such abbreviations can only be suggested by experience and practice, and the beginner should not waste his time in seeking for them.

213. Solve $\frac{12}{x}+\frac{8}{y}=8, \quad \frac{27}{x}-\frac{12}{y}=3.$

If we cleared these equations of fractions they would involve the product xy of the unknown quantities; and thus strictly they do not belong to the present Chapter. But they may be solved by the methods already given, as we shall now shew. For multiply the first equation by 3 and the second by 2, and add; thus

$$\frac{36}{x}+\frac{24}{y}+\frac{54}{x}-\frac{24}{y}=24+6;$$

that is, $$\frac{36}{x}+\frac{54}{x}=30;$$

that is, $$\frac{90}{x}=30;$$

therefore $$90=30x;$$

therefore $$x=3.$$

Substitute the value of x in the first equation; thus

$$\frac{12}{3}+\frac{8}{y}=8;$$

therefore $$\frac{8}{y}=8-4=4;$$

therefore $$8=4y;$$

therefore $$y=2.$$

214. Solve $a^2x+b^2y=c^2, \quad ax+by=c.$

Here x and y are supposed to denote *unknown* quantities, while the other letters are supposed to denote *known* quantities.

Multiply the second equation by b, and subtract it from the first; thus

$$a^2x+b^2y-abx-b^2y=c^2-bc;$$

that is, $$a(a-b)x=c(c-b);$$

therefore $$x=\frac{c(c-b)}{a(a-b)}.$$

Substitute this value of x in the second equation; thus

$$\frac{ac(c-b)}{a(a-b)}+by=c;$$

therefore $$by=c-\frac{c(c-b)}{a-b}=\frac{c(a-b)-c(c-b)}{a-b}=\frac{c(a-c)}{a-b};$$

therefore $$y=\frac{c(a-c)}{b(a-b)}=\frac{c(c-a)}{b(b-a)}.$$

Or the value of y might be found in the same way as that of x was found.

EXAMPLES. XXIII.

1. $3x-4y=2, \qquad 7x-9y=7.$
2. $7x-5y=24, \qquad 4x-3y=11.$
3. $3x+2y=32, \qquad 20x-3y=1.$
4. $11x-7y=37, \qquad 8x+9y=41.$
5. $7x+5y=60, \qquad 13x-11y=10.$
6. $6x-7y=42, \qquad 7x-6y=75.$
7. $10x+9y=290, \qquad 12x-11y=130.$
8. $3x-4y=18, \qquad 3x+2y=0.$
9. $4x-\frac{y}{2}=11, \qquad 2x-3y=0.$

10. $\frac{x}{3}+3y=7, \quad \frac{4x-2}{5}=3y-4.$

11. $6x-5y=1, \quad 7x-4y=8\frac{1}{2}.$

12. $2x+\frac{y-2}{5}=21, \quad 4y+\frac{x-4}{6}=29.$

13. $\frac{3x}{19}+5y=13, \quad 2x+\frac{4-7y}{2}=33.$

14. $\frac{x}{7}+\frac{y}{14}=10\frac{1}{2}, \quad 2x-y=7.$

15. $\frac{x+y}{3}+\frac{y-x}{2}=9, \quad \frac{x}{2}+\frac{x+y}{9}=5.$

16. $\frac{3x}{4}-\frac{2y}{3}=1, \quad \frac{7x}{3}+\frac{5y}{6}=6.$

17. $\frac{x+y}{3}+x=15, \quad \frac{x-y}{5}+y=6.$

18. $\frac{7x}{6}+\frac{5y}{3}=34, \quad \frac{7x}{8}+\frac{3y}{4}=\frac{5y}{8}+12.$

19. $\frac{x+y}{8}+\frac{x-y}{6}=5, \quad \frac{x+y}{4}-\frac{x-y}{3}=10.$

20. $\frac{2x}{3}+\frac{3y}{2}=16\frac{1}{6}, \quad \frac{3x}{2}-\frac{2y}{3}=16\frac{1}{6}.$

21. $\frac{x-1}{8}+\frac{y-2}{5}=2, \quad 2x+\frac{2y-5}{3}=21.$

22. $\frac{7x}{4}+\frac{5y}{8}=20, \quad \frac{3x}{5}+\frac{7y}{4}=2x-7.$

23. $\frac{2x+3y}{5}=10-\frac{y}{3}, \quad \frac{4y-3x}{6}=\frac{3x}{4}+1.$

24. $\frac{1-3x}{7}+\frac{3y-1}{5}=2, \quad \frac{3x+y}{11}+y=9.$

25. $2(2x+3y)=3(2x-3y)+10,$
$4x-3y=4(6y-2x)+3.$

26. $3x+9y=2{\cdot}4,\qquad {\cdot}21x-{\cdot}06y={\cdot}03.$

27. ${\cdot}3x+{\cdot}125y=x-6,\qquad 3x-{\cdot}5y=28-{\cdot}25y.$

28. ${\cdot}08x-{\cdot}21y={\cdot}33,\qquad {\cdot}12x+{\cdot}7y=3{\cdot}54.$

29. $\frac{9}{x}-\frac{4}{y}=1,\qquad \frac{18}{x}+\frac{20}{y}=16.$

30. $x-4y=7,\qquad \frac{x}{3y}+\frac{11}{10}=\frac{4x-5y}{5y}.$

31. $\frac{x+1}{y-1}-\frac{x-1}{y}=\frac{6}{y},\qquad x-y=1.$

32. $4x+y=11,\qquad \frac{y}{5x}=\frac{7x-y}{3x}-\frac{23}{15}.$

33. $\frac{x+\frac{y}{2}-3}{x-5}+7=0,\qquad \frac{3y-10(x-1)}{6}+\frac{x-y}{4}+1=0.$

34. $\frac{x}{a}+\frac{y}{b}=2,\qquad bx-ay=0.$

35. $x+y=a+b,\qquad bx+ay=2ab.$

36. $\frac{x}{a}+\frac{y}{b}=1,\qquad \frac{x}{b}+\frac{y}{a}=1.$

37. $(a+c)x-by=bc,\qquad x+y=a+b.$

38. $\frac{x}{a}+\frac{y}{b}=c,\qquad \frac{x}{b}-\frac{y}{a}=0.$

39. $x+y=c,\qquad ax-by=c(a-b).$

40. $a(x+y)+b(x-y)=1,\qquad a(x-y)+b(x+y)=1.$

41. $\frac{x-a}{b}+\frac{y-b}{a}=0,\qquad \frac{x+y-b}{a}+\frac{x-y-a}{b}=0.$

42. $(a+b)x-(a-b)y=4ab,$
$(a-b)x+(a+b)y=2a^2-2b^2.$

43. $\frac{x}{a+b}+\frac{y}{a-b}=2a,\qquad \frac{x-y}{2ab}=\frac{x+y}{a^2+b^2}.$

44. $(a+h)x+(b-h)y=c,\qquad (b+k)x+(a-k)y=c.$

XXIV. *Simultaneous equations of the first degree with more than two unknown quantities.*

215. If there be three simple equations containing three unknown quantities, we can deduce from two of the equations an equation which contains only two of the unknown quantities, by the methods of the preceding Chapter; then from the third given equation, and either of the former two, we can deduce another equation which contains the same two unknown quantities. We have thus two equations containing two unknown quantities, and therefore the values of these unknown quantities may be found by the methods of the preceding Chapter. By substituting these values in one of the given equations, the value of the remaining unknown quantity may be found.

216. Solve $7x+3y-2z=16$ .... (1),

$$2x+5y+3z=39 \text{ .... (2)},$$

$$5x-y+5z=31 \text{ .... (3)}.$$

For convenience of reference the equations are numbered (1), (2), (3); and this numbering is continued as we proceed with the solution.

Multiply (1) by 3, and multiply (2) by 2; thus

$$21x+9y-6z=48,$$

$$4x+10y+6z=78;$$

therefore, by addition,

$$25x+19y=126 \text{ .... (4)}.$$

Multiply (1) by 5, and multiply (3) by 2; thus

$$35x+15y-10z=80,$$

$$10x-2y+10z=62;$$

therefore, by addition,

$$45x+13y=142 \text{ .... (5)}.$$

We have now to find the values of x and y from (4) and (5).

Multiply (4) by 9, and multiply (5) by 5; thus

$$225x + 171y = 1134,$$

$$225x + 65y = 710;$$

therefore, by subtraction,

$$106y = 424;$$

therefore $$y = 4.$$

Substitute the value of y in (4); thus

$$25x + 76 = 126;$$

therefore $$25x = 126 - 76 = 50;$$

therefore $$x = 2.$$

Substitute the values of x and y in (1); thus

$$14 + 12 - 2z = 16;$$

therefore $$10 = 2z;$$

therefore $$z = 5.$$

217. Solve $\frac{1}{x} + \frac{2}{y} - \frac{3}{z} = 1$ .... (1),

$$\frac{5}{x} + \frac{4}{y} + \frac{6}{z} = 24 \text{ .... (2)},$$

$$\frac{7}{x} - \frac{8}{y} + \frac{9}{z} = 14 \text{ .... (3)}.$$

Multiply (1) by 2, and add the result to (2); thus

$$\frac{2}{x} + \frac{4}{y} - \frac{6}{z} + \frac{5}{x} + \frac{4}{y} + \frac{6}{z} = 2 + 24;$$

that is, $$\frac{7}{x} + \frac{8}{y} = 26 \text{ .... (4)}.$$

Multiply (1) by 3, and add the result to (3); thus

$$\frac{3}{x}+\frac{6}{y}-\frac{9}{z}+\frac{7}{x}-\frac{8}{y}+\frac{9}{z}=3+14;$$

that is, $$\frac{10}{x}-\frac{2}{y}=17 \quad \ldots\ldots\ldots (5).$$

Multiply (5) by 4, and add the result to (4); thus

$$\frac{40}{x}-\frac{8}{y}+\frac{7}{x}+\frac{8}{y}=68+26;$$

that is, $$\frac{47}{x}=94;$$

therefore $$47=94x;$$

therefore $$x=\frac{47}{94}=\frac{1}{2}.$$

Substitute the value of x in (5); thus

$$20-\frac{2}{y}=17;$$

therefore $$\frac{2}{y}=20-17=3;$$

therefore $$y=\frac{2}{3}.$$

Substitute the values of x and y in (1); thus

$$2+3-\frac{3}{z}=1;$$

therefore $$\frac{3}{z}=4;$$

therefore $$z=\frac{3}{4}.$$

218. Solve

$$\frac{x}{a}+\frac{y}{b}=3 \;\ldots\ldots\; (1),$$

$$\frac{y}{b}+\frac{z}{c}=5 \;\ldots\ldots\; (2),$$

$$\frac{x}{a}+\frac{z}{c}=4 \;\ldots\ldots\; (3).$$

Subtract (1) from (2); thus

$$\frac{y}{b}+\frac{z}{c}-\frac{x}{a}-\frac{y}{b}=5-3;$$

that is,
$$\frac{z}{c}-\frac{x}{a}=2 \;\ldots\ldots\ldots\; (4).$$

By subtracting (4) from (3) we obtain

$$\frac{2x}{a}=2;$$

therefore $\frac{x}{a}=1$; therefore $x=a$.

By adding (4) to (3) we obtain

$$\frac{2z}{c}=6;$$

therefore $\frac{z}{c}=3$; therefore $z=3c$.

By substituting the value of x in (1) we find that $y=2b$.

219. In a similar manner we may proceed if the number of equations and unknown quantities should exceed three.

EXAMPLES. XXIV.

1. $x+3y+2z=11$, $2x+y+3z=14$, $3x+2y+z=11$.

2. $5x-6y+4z=15$, $7x+4y-3z=19$, $2x+y+6z=46$.

3. $4x-5y+z=6$, $7x-11y+2z=9$, $x+y+3z=12$.

4. $7x-3y=30$, $9y-5z=34$, $x+y+z=33$.

5. $3x-y+z=17$, $5x+3y-2z=10$, $7x+4y-5z=3$.

6. $x+y+z=5$, $3x-5y+7z=75$, $9x-11z+10=0$.

7. $x+2y+3z=6$, $2x+4y+2z=8$, $3x+2y+8z=101$.

8. $\dfrac{6y-4x}{3z-7}=1$, $\dfrac{5z-x}{2y-3z}=1$, $\dfrac{y-2z}{3y-2x}=1$.

9. $\dfrac{x+2y}{7}=\dfrac{3y+4z}{8}=\dfrac{5x+6z}{9}$, $x+y-z=126$.

10. $\dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{6}$, $\dfrac{1}{y}+\dfrac{1}{z}=3\tfrac{5}{6}$, $\dfrac{4}{x}+\dfrac{3}{y}=\dfrac{4}{z}$.

11. $y+z=a$, $z+x=b$, $x+y=c$.

12. $x+y+z=a+b+c$, $x+a=y+b=z+c$.

13. $y+z-x=a$, $z+x-y=b$, $x+y-z=c$.

14. $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$, $\dfrac{x}{a}+\dfrac{y}{c}+\dfrac{z}{b}=1$, $\dfrac{x}{b}+\dfrac{y}{a}+\dfrac{z}{c}=1$.

15. $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=3$, $\dfrac{a}{x}+\dfrac{b}{y}-\dfrac{c}{z}=1$, $\dfrac{2a}{x}-\dfrac{b}{y}-\dfrac{c}{z}=0$.

16. $v+x+y+z=14$,
$2v+x=2y+z-2$,
$3v-x+2y+2z=19$,
$\dfrac{v}{3}+\dfrac{x}{4}+\dfrac{y}{5}+\dfrac{z}{2}=4$.

XXV. *Problems which lead to simultaneous equations of the first degree with more than one unknown quantity.*

220. We shall now solve some problems which lead to simultaneous equations of the first degree with more than one unknown quantity.

Find the fraction which becomes equal to $\frac{2}{3}$ when the numerator is increased by 2, and equal to $\frac{4}{7}$ when the denominator is increased by 4.

Let x denote the numerator, and y the denominator of the required fraction; then, by supposition,

$$\frac{x+2}{y}=\frac{2}{3}, \qquad \frac{x}{y+4}=\frac{4}{7}.$$

Clear the equations of fractions; thus we obtain

$$3x-2y=-6 \text{ .... } (1),$$
$$7x-4y=\ 16 \text{ .... } (2).$$

Multiply (1) by 2, and subtract it from (2); thus

$$7x-4y-6x+4y=16+12;$$

that is, $$x=28.$$

Substitute the value of x in (1); thus

$$84-2y=-6;$$

therefore $2y=90$; therefore $y=45$.

Hence the required fraction is $\frac{28}{45}$.

221. A sum of money was divided equally among a certain number of persons; if there had been six more, each would have received two shillings less than he did; and if there had been three fewer, each would have received two shillings more than he did: find the number of persons, and what each received.

Let x denote the number of persons, and y the number of shillings which each received. Then xy is the number of shillings in the sum of money which is divided; and, by supposition,

$$(x+6)(y-2)=xy \text{ .... } (1),$$

$$(x-3)(y+2)=xy \text{ .... } (2).$$

From (1) we obtain

$$xy+6y-2x-12=xy;$$

therefore $$6y-2x=12 \text{ .... } (3).$$

From (2) we obtain

$$xy+2x-3y-6=xy;$$

therefore $$2x-3y=6 \text{ .... } (4).$$

From (3) and (4), by addition, $3y=18$; therefore $y=6$.

Substitute the value of y in (4); thus

$$2x-18=6;$$

therefore $2x=24$; therefore $x=12$.

Thus there were 12 persons, and each received 6 shillings.

222. A certain number of two digits is equal to five times the sum of its digits; and if nine be added to the number the digits are reversed: find the number.

Let x denote the digit in the tens' place, and y the digit in the units' place. Then the number is $10x+y$; and, by supposition, the number is equal to five times the sum of its digits; therefore

$$10x+y=5(x+y) \text{ .... } (1).$$

If nine be added to the number its digits are reversed, that is, we obtain the number $10y+x$; therefore

$$10x+y+9=10y+x \text{ .... } (2).$$

From (1) we obtain

$$5x=4y \text{ .... } (3).$$

From (2) we obtain $9x+9=9y$; therefore $x+1=y$.

Substitute for y in (3); thus

$$5x = 4x + 4;$$

therefore $$x = 4.$$

Then from (3) we obtain $y = 5$.

Hence the required number is 45.

223. A railway train after travelling an hour is detained 24 minutes, after which it proceeds at six-fifths of its former rate, and arrives 15 minutes late. If the detention had taken place 5 miles further on, the train would have arrived 2 minutes later than it did. Find the original rate of the train, and the distance travelled.

Let $5x$ denote the number of miles per hour at which the train originally travelled, and let y denote the number of miles in the whole distance travelled. Then $y - 5x$ will denote the number of miles which remain to be travelled after the detention. At the original rate of the train this distance would be travelled in $\frac{y-5x}{5x}$ hours; at the increased rate it will be travelled in $\frac{y-5x}{6x}$ hours. Since the train is detained 24 minutes, and yet is only 15 minutes late at its arrival, it follows that the remainder of the journey is performed in 9 minutes less than it would have been if the rate had not been increased. And 9 minutes is $\frac{9}{60}$ of an hour; therefore

$$\frac{y-5x}{6x} = \frac{y-5x}{5x} - \frac{9}{60} \text{ .... (1).}$$

If the detention had taken place 5 miles further on, there would have been $y - 5x - 5$ miles left to be travelled. Thus we shall find that

$$\frac{y-5x-5}{6x} = \frac{y-5x-5}{5x} - \frac{7}{60} \text{ .... (2).}$$

Subtract (2) from (1); thus

$$\frac{5}{6x} = \frac{5}{5x} - \frac{2}{60};$$

therefore $50 = 60 - 2x$;

therefore $2x = 10$; therefore $x = 5$.

Substitute this value of x in (1), and it will be found by solving the equation that $y = 47\frac{1}{2}$.

224. A, B, and C can together perform a piece of work in 30 days; A and B can together perform it in 32 days; and B and C can together perform it in 120 days: find the time in which each alone could perform the work.

Let x denote the number of days in which A alone could perform it, y the number of days in which B alone could perform it, z the number of days in which C alone could perform it. Then we have

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{30} \text{ .... } (1),$$

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{32} \text{ .... } (2),$$

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{120} \text{ .... } (3).$$

Subtract (2) from (1); thus

$$\frac{1}{z} = \frac{1}{30} - \frac{1}{32} = \frac{1}{480}.$$

Subtract (3) from (1); thus

$$\frac{1}{x} = \frac{1}{30} - \frac{1}{120} = \frac{1}{40}.$$

Therefore $x = 40$, and $z = 480$; and by substitution in any of the given equations we shall find that $y = 160$.

225. We may observe that a problem may often be solved in various ways, and with the aid of more or fewer letters to represent the unknown quantities. Thus, to take a very simple example, suppose we have to find two

numbers such that one is two-thirds of the other, and their sum is 100.

We may proceed thus. Let x denote the greater number, and y the less number; then we have

$$y=\frac{2x}{3}, \quad x+y=100.$$

Or we may proceed thus. Let x denote the greater number, then $100-x$ will denote the less number; therefore

$$100-x=\frac{2x}{3}.$$

Or we may proceed thus. Let $3x$ denote the greater number, then $2x$ will denote the less number; therefore

$$2x+3x=100.$$

By completing any of these processes we shall find that the required numbers are 60 and 40.

The student may accordingly find that he can solve some of the examples at the end of the present Chapter, with the aid of only one letter to denote an unknown quantity; and, on the other hand, some of the examples at the end of Chapter XXII. may appear to him most naturally solved with the aid of two letters. As a general rule it may be stated that the employment of a larger number of unknown quantities renders the work longer, but at the same time allows the successive steps to be more readily followed; and thus is more suitable for beginners.

The beginner will find it a good exercise to solve the example given in Art. 204 with the aid of four letters to represent the four unknown quantities which are required.

EXAMPLES. XXV.

1. If A's money were increased by 36 shillings he would have three times as much as B; and if B's money were diminished by 5 shillings he would have half as much as A: find the sum possessed by each.

2. Find two numbers such that the first with half the second may make 20, and also that the second with a third of the first may make 20.

3. If B were to give £25 to A they would have equal sums of money; if A were to give £22 to B the money of B would be double that of A: find the money which each actually has.

4. Find two numbers such that half the first with a third of the second may make 32, and that a fourth of the first with a fifth of the second may make 18.

5. A person buys 8 lbs. of tea and 3 lbs. of sugar for £1. 2*s.*; and at another time he buys 5 lbs. of tea and 4 lbs. of sugar for 15*s.* 2*d.*: find the price of tea and sugar per lb.

6. Seven years ago A was three times as old as B was; and seven years hence A will be twice as old as B will be: find their present ages.

7. Find the fraction which becomes equal to $\frac{1}{3}$ when the numerator is increased by 1, and equal to $\frac{1}{4}$ when the denominator is increased by 1.

8. A certain fishing rod consists of two parts; the length of the upper part is to the length of the lower as 5 to 7; and 9 times the upper part together with 13 times the lower part exceed 11 times the whole rod by 36 inches: find the lengths of the two parts.

9. A person spends half-a-crown in apples and pears, buying the apples at 4 a penny, and the pears at 5 a penny; he sells half his apples and one-third of his pears for 13 pence, which was the price at which he bought them: find how many apples and how many pears he bought.

10. A wine merchant has two sorts of wine, a better and a worse; if he mixes them in the proportion of two quarts of the better sort with three of the worse, the mixture will be worth 1*s.* 9*d.* a quart; but if he mixes them in the proportion of seven quarts of the better sort with eight of the worse, the mixture will be worth 1*s.* 10*d.* a quart: find the price of a quart of each sort.

11. A farmer sold to one person 30 bushels of wheat, and 40 bushels of barley for £13. 10*s.*; to another person he sold 50 bushels of wheat and 30 bushels of barley for £17: find the price of wheat and barley per bushel.

12. A farmer has 28 bushels of barley at 2*s.* 4*d.* a bushel: with these he wishes to mix rye at 3*s.* a bushel, and wheat at 4*s.* a bushel, so that the mixture may consist of 100 bushels, and be worth 3*s.* 4*d.* a bushel: find how many bushels of rye and wheat he must take.

13. *A* and *B* lay a wager of 10 shillings; if *A* loses he will have as much as *B* will then have; if *B* loses he will have half of what *A* will then have: find the money of each.

14. If the numerator of a certain fraction be increased by 1, and the denominator be diminished by 1, the value will be 1; if the numerator be increased by the denominator, and the denominator diminished by the numerator, the value will be 4: find the fraction.

15. A number of posts are placed at equal distances in a straight line. If to twice the number of them we add the distance between two consecutive posts, expressed in feet, the sum is 68. If from four times the distance between two consecutive posts, expressed in feet, we subtract half the number of posts, the remainder is 68. Find the distance between the extreme posts.

16. A gentleman distributing money among some poor men found that he wanted 10 shillings, in order to be able to give 5 shillings to each man; therefore he gives to each man 4 shillings only, and finds that he has 5 shillings left: find the number of poor men and of shillings.

17. A certain company in a tavern found, when they came to pay their bill, that if there had been three more persons to pay the same bill, they would have paid one shilling each less than they did; and if there had been two fewer persons they would have paid one shilling each more than they did: find the number of persons and the number of shillings each paid.

18. There is a certain rectangular floor, such that if it had been two feet broader, and three feet longer, it would have been sixty-four square feet larger; but if it had been three feet broader, and two feet longer, it would have been sixty-eight square feet larger: find the length and breadth of the floor.

19. A certain number of two digits is equal to four

times the sum of its digits; and if 18 be added to the number the digits are reversed: find the number.

20. Two digits which form a number change places on the addition of 9; and the sum of the two numbers is 33: find the digits.

21. When a certain number of two digits is doubled, and increased by 36, the result is the same as if the number had been reversed, and doubled, and then diminished by 36; also the number itself exceeds four times the sum of its digits by 3: find the number.

22. Two passengers have together 5 cwt. of luggage, and are charged for the excess above the weight allowed 5*s.* 2*d.* and 9*s.* 10*d.* respectively; if the luggage had all belonged to one of them he would have been charged 19*s.* 2*d.*: find how much luggage each passenger is allowed without charge.

23. A and B ran a race which lasted 5 minutes; B had a start of 20 yards; but A ran 3 yards while B was running 2, and won by 30 yards: find the length of the course and the speed of each.

24. A and B have each a certain number of counters; A gives to B as many as B has already, and B returns back again to A as many as A has left; A gives to B as many as B has left, and B returns to A as many as A has left; each of them has now sixteen counters: find how many each had at first.

25. A and B can together perform a certain work in 30 days; at the end of 18 days however B is called off and A finishes it alone in 20 more days: find the time in which each could perform the work alone.

26. A, B, and C can drink a cask of beer in 15 days; A and B together drink four-thirds of what C does; and C drinks twice as much as A: find the time in which each alone could drink the cask of beer.

27. A cistern holding 1200 gallons is filled by three pipes A, B, C together in 24 minutes. The pipe A requires 30 minutes more than C to fill the cistern; and 10 gallons less run through C per minute than through A and B together. Find the time in which each pipe alone would fill the cistern.

28. *A* and *B* run a mile. At the first heat *A* gives *B* a start of 20 yards, and beats him by 30 seconds. At the second heat *A* gives *B* a start of 32 seconds, and beats him by $9\frac{5}{11}$ yards. Find the rate per hour at which *A* runs.

29. *A* and *B* are two towns situated 24 miles apart, on the same bank of a river. A man goes from *A* to *B* in 7 hours, by rowing the first half of the distance, and walking the second half. In returning he walks the first half at three-fourths of his former rate, but the stream being with him he rows at double his rate in going; and he accomplishes the whole distance in 6 hours. Find his rates of walking and rowing.

30. A railway train after travelling an hour is detained 15 minutes, after which it proceeds at three-fourths of its former rate, and arrives 24 minutes late. If the detention had taken place 5 miles further on, the train would have arrived 3 minutes sooner than it did. Find the original rate of the train and the distance travelled.

31. The time which an express train takes to travel a journey of 120 miles is to that taken by an ordinary train as 9 is to 14. The ordinary train loses as much time in stoppages as it would take to travel 20 miles without stopping. The express train only loses half as much time in stoppages as the ordinary train, and it also travels 15 miles an hour quicker. Find the rate of each train.

32. Two trains, 92 feet long and 84 feet long respectively, are moving with uniform velocities on parallel rails; when they move in opposite directions they are observed to pass each other in one second and a half; but when they move in the same direction the faster train is observed to pass the other in six seconds: find the rate at which each train moves.

33. A railroad runs from *A* to *C*. A goods' train starts from *A* at 12 o'clock, and a passenger train at 1 o'clock. After going two-thirds of the distance the goods' train breaks down, and can only travel at three-fourths of its former rate. At 40 minutes past 2 o'clock a collision occurs, 10 miles from *C*. The rate of the passenger train is double the diminished rate of the goods' train. Find the distance from *A* to *C*, and the rates of the trains.

34. A certain sum of money was divided between *A*, *B*, and *C*, so that *A*'s share exceeded four-sevenths of the shares of *B* and *C* by £30; also *B*'s share exceeded three-eighths of the shares of *A* and *C* by £30; and *C*'s share exceeded two-ninths of the shares of *A* and *B* by £30. Find the share of each person.

35. *A* and *B* working together can earn 40 shillings in 6 days; *A* and *C* together can earn 54 shillings in 9 days; and *B* and *C* together can earn 80 shillings in 15 days: find what each man can earn alone per day.

36. A certain number of sovereigns, shillings, and sixpences amount to £8. 6*s*. 6*d*. The amount of the shillings is a guinea less than that of the sovereigns, and a guinea and a half more than that of the sixpences. Find the number of each coin.

37. *A* and *B* can perform a piece of work together in 48 days; *A* and *C* in 30 days; and *B* and *C* in $26\frac{2}{3}$ days: find the time in which each could perform the work alone.

38. There is a certain number of three digits which is equal to 48 times the sum of its digits, and if 198 be subtracted from the number the digits will be reversed; also the sum of the extreme digits is equal to twice the middle digit: find the number.

39. A man bought 10 bullocks, 120 sheep, and 46 lambs. The price of 3 sheep is equal to that of 5 lambs. A bullock, a sheep, and a lamb together cost a number of shillings greater by 300 than the whole number of animals bought; and the whole sum spent was £468. 6*s*. Find the price of a bullock, a sheep, and a lamb respectively.

40. A farmer sold at a market 100 head of stock consisting of horses, oxen, and sheep, so that the whole realised £2. 7*s*. per head; while a horse, an ox, and a sheep were sold for £22, £12. 10*s*., and £1. 10*s*. respectively. Had he sold one-fourth the number of oxen, and 25 more sheep than he did, the amount received would have been still the same. Find the number of horses, oxen, and sheep, respectively which were sold.

XXVI. *Quadratic Equations.*

226. A quadratic equation is an equation which contains the *square* of the unknown quantity, but no higher power.

227. A *pure* quadratic equation is one which contains *only* the square of the unknown quantity. An *adfected* quadratic equation is one which contains the first power of the unknown quantity as well as its square. Thus, for example, $2x^2=50$ is a *pure* quadratic equation; and $2x^2-7x+3=0$ is an *adfected* quadratic equation.

228. The following is the Rule for solving a pure quadratic equation. *Find the value of the square of the unknown quantity by the Rule for solving a simple equation; then, by extracting the square root, the values of the unknown quantity are found.*

For example, solve $\frac{x^2-13}{3}+\frac{x^2-5}{10}=6$.

Clear of fractions by multiplying by 30; thus

$$10(x^2-13)+3(x^2-5)=180;$$

therefore $$13x^2=180+130+15=325;$$

therefore $$x^2=\frac{325}{13}=25;$$

extract the square root, thus $x=\pm 5$.

In this example, we find by the Rule for solving a simple equation, that x^2 is equal to 25; therefore x must be such a number, that if multiplied into itself the product is 25. That is to say, x must be a square root of 25. In Arithmetic 5 is the square root of 25; in Algebra we may consider either 5 or -5 as a square root of 25, since, by the *Rule of Signs* $-5\times-5=5\times5$. Hence x may have either of the values 5 or -5, and the equation will be satisfied. This we denote thus, $x=\pm5$.

229. We proceed to the solution of adfected quadratics.

If we multiply $x+\frac{a}{2}$ by itself we obtain

$$\left(x+\frac{a}{2}\right)\left(x+\frac{a}{2}\right)=x^2+2\frac{ax}{2}+\frac{a^2}{4}=x^2+ax+\frac{a^2}{4};$$

thus $x^2+ax+\frac{a^2}{4}$ is a *perfect square*, for it is the square of $x+\frac{a}{2}$. Hence x^2+ax is rendered a perfect square by the addition of $\frac{a^2}{4}$, that is, *by the addition of the square of half the coefficient of x*. This fact is the essential part of the solution of an adfected quadratic equation, and we shall now give some examples of it.

x^2+6x; here half the coefficient of x is 3; add 3^2, and we obtain x^2+6x+3^2, that is $(x+3)^2$.

x^2-5x; here half the coefficient of x is $-\frac{5}{2}$; add $\left(-\frac{5}{2}\right)^2$, that is $\left(\frac{5}{2}\right)^2$, and we obtain $x^2-5x+\left(\frac{5}{2}\right)^2$, that is $\left(x-\frac{5}{2}\right)^2$.

$x^2+\frac{4x}{5}$; here half the coefficient of x is $\frac{2}{5}$; add $\left(\frac{2}{5}\right)^2$, and we obtain $x^2+\frac{4x}{5}+\left(\frac{2}{5}\right)^2$, that is $\left(x+\frac{2}{5}\right)^2$.

$x^2-\frac{3x}{4}$; here half the coefficient of x is $-\frac{3}{8}$; add $\left(-\frac{3}{8}\right)^2$, that is $\left(\frac{3}{8}\right)^2$, and we obtain $x^2-\frac{3x}{4}+\left(\frac{3}{8}\right)^2$, that is $\left(x-\frac{3}{8}\right)^2$.

The process here exemplified is called *completing the square.*

230. The following is the Rule for solving an adfected quadratic equation. *By transposition and reduction arrange the equation so that the terms which involve the unknown quantity are alone on one side, and the coefficient of* x^2 *is* $+1$*; add to each side of the equation the square of half the coefficient of* x*, and then extract the square root of each side.*

It will be seen from the examples which we shall now solve that the above rule leads us to a point from which we can immediately obtain the values of the unknown quantity.

231. Solve $x^2-10x+24=0.$

By transposition, $x^2-10x=-24;$

add $\left(\frac{10}{2}\right)^2$, $x^2-10x+5^2=-24+25=1;$

extract the square root, $x-5=\pm 1;$

transpose, $x=5\pm 1=5+1$ or $5-1;$

hence $x=6$ or $4.$

It is easy to verify that either of these values satisfies the proposed equation; and it will be useful for the student thus to verify his results.

232. Solve $3x^2-4x-55=0.$

By transposition, $3x^2-4x=55;$

divide by 3, $x^2-\frac{4x}{3}=\frac{55}{3};$

add $\left(\frac{2}{3}\right)^2$, $x^2-\frac{4x}{3}+\left(\frac{2}{3}\right)^2=\frac{55}{3}+\frac{4}{9}=\frac{169}{9};$

extract the square root, $x-\frac{2}{3}=\pm\frac{13}{3};$

transpose, $x=\frac{2}{3}\pm\frac{13}{3}=5$ or $-\frac{11}{3}.$

233. Solve $2x^2 + 3x - 35 = 0$.

By transposition, $2x^2 + 3x = 35$;

divide by 2, $x^2 + \frac{3x}{2} = \frac{35}{2}$;

add $\left(\frac{3}{4}\right)^2$, $x^2 + \frac{3x}{2} + \left(\frac{3}{4}\right)^2 = \frac{35}{2} + \frac{9}{16} = \frac{289}{16}$;

extract the square root, $x + \frac{3}{4} = \pm \frac{17}{4}$;

transpose, $x = -\frac{3}{4} \pm \frac{17}{4} = \frac{7}{2}$ or -5.

234. Solve $x^2 - 4x - 1 = 0$.

By transposition, $x^2 - 4x = 1$;

add 2^2, $x^2 - 4x + 2^2 = 1 + 4 = 5$;

extract the square root, $x - 2 = \pm \sqrt{5}$;

transpose, $x = 2 \pm \sqrt{5}$.

Here the square root of 5 cannot be found exactly; but we can find by Arithmetic an approximate value of it to any assigned degree of accuracy, and thus obtain the values of x to any assigned degree of accuracy.

235. In the examples hitherto solved we have found two different roots of a quadratic equation; in some cases however we shall find really only one root. Take, for example, the equation $x^2 - 14x + 49 = 0$; by extracting the square root we have $x - 7 = 0$, therefore $x = 7$. It is however found convenient in such a case to say that the quadratic equation has *two equal roots.*

236. Solve $x^2-6x+13=0.$

By transposition, $x^2-6x=-13;$

add 3^2, $x^2-6x+3^2=-13+9=-4.$

If we try to extract the square root we have

$$x-3=\pm\sqrt{-4}.$$

But -4 can have no square root, exact or approximate, because any number, whether positive or negative, if multiplied by itself, gives a positive result. In this case the quadratic equation has no real root; and this is sometimes expressed by saying that the roots are *imaginary* or *impossible*.

237. Solve $\frac{1}{2(x-1)}+\frac{3}{x^2-1}=\frac{1}{4}.$

Here we first clear of fractions by multiplying by $4(x^2-1)$, which is the least common multiple of the denominators.

Thus $2(x+1)+12=x^2-1.$

By transposition, $x^2-2x=15;$

add 1^2, $x^2-2x+1=15+1=16;$

extract the square root, $x-1=\pm 4;$

therefore $x=1\pm 4=5$ or -3.

238. Solve $\frac{2x}{15}+\frac{3x-50}{3(10+x)}=\frac{12x+70}{190}.$

Multiply by 570, which is the least common multiple of 15 and 190; thus

$$76x+\frac{190(3x-50)}{10+x}=3(12x+70);$$

therefore $\frac{190(3x-50)}{10+x}=210-40x;$

therefore $190(3x-50)=(210-40x)(10+x);$

that is, $570x - 9500 = 2100 - 190x - 40x^2$;

therefore $40x^2 + 760x = 11600$;

therefore $x^2 + 19x = 290$;

add $\left(\frac{19}{2}\right)^2$, $x^2 + 19x + \left(\frac{19}{2}\right)^2 = 290 + \frac{361}{4} = \frac{1521}{4}$;

extract the square root, $x + \frac{19}{2} = \pm \frac{39}{2}$;

therefore $x = -\frac{19}{2} \pm \frac{39}{2} = 10$ or -29.

239. Solve $\frac{x+3}{x+2} + \frac{x-3}{x-2} = \frac{2x-3}{x-1}$.

Clear of fractions; thus

$$(x+3)(x-2)(x-1) + (x-3)(x+2)(x-1) = (2x-3)(x+2)(x-2);$$

that is, $x^3 - 7x + 6 + x^3 - 2x^2 - 5x + 6 = 2x^3 - 3x^2 - 8x + 12$;

that is, $2x^3 - 2x^2 - 12x + 12 = 2x^3 - 3x^2 - 8x + 12$;

therefore $x^2 - 4x = 0$;

add 2^2, $x^2 - 4x + 2^2 = 4$;

extract the square root, $x - 2 = \pm 2$,

therefore $x = 2 \pm 2 = 4$ or 0.

We have given the last three lines in order to complete the solution of the equation in the same manner as in the former examples; but the results may be obtained more simply. For the equation $x^2 - 4x = 0$ may be written $(x-4)x = 0$; and in this form it is sufficiently obvious that we must have either $x - 4 = 0$, or $x = 0$, that is, $x = 4$ or 0.

The student will observe that in this example $2x^3$ is found on both sides of the equation, after we have cleared of fractions; accordingly it can be removed by subtraction, and so the equation remains a quadratic equation.

240. *Every quadratic equation can be put in the form* $x^2+px+q=0$, *where* p *and* q *represent some known numbers, whole or fractional, positive or negative.*

For a quadratic equation, by definition, contains no power of the unknown quantity higher than the second. Let all the terms be brought to one side, and, if necessary, change the signs of all the terms so that the coefficient of the square of the unknown quantity may be a positive number; then divide every term by this coefficient, and the equation takes the assigned form.

For example, suppose $7x-4x^2=5$. Here we have

$$7x-4x^2-5=0;$$

therefore $$4x^2-7x+5=0;$$

therefore $$x^2-\frac{7x}{4}+\frac{5}{4}=0.$$

Thus in this example we have $p=-\frac{7}{4}$ and $q=\frac{5}{4}$.

241. Solve $$x^2+px+q=0.$$

By transposition, $$x^2+px=-q;$$

add $\left(\frac{p}{2}\right)^2$, $$x^2+px+\left(\frac{p}{2}\right)^2=-q+\frac{p^2}{4}=\frac{p^2-4q}{4};$$

extract the square root, $$x+\frac{p}{2}=\pm\frac{\sqrt{(p^2-4q)}}{2};$$

therefore $$x=-\frac{p}{2}\pm\frac{\sqrt{(p^2-4q)}}{2}=\frac{-p\pm\sqrt{(p^2-4q)}}{2}.$$

242. We have thus obtained a *general formula* for the roots of the quadratic equation $x^2+px+q=0$, namely, that x must be equal to

$$\frac{-p+\sqrt{(p^2-4q)}}{2} \text{ or to } \frac{-p-\sqrt{(p^2-4q)}}{2}.$$

We shall now deduce from this general formula some very important inferences, which will hold for any quadratic equation, by Art. 240.

243. *A quadratic equation cannot have more than two roots.*

For we have seen that the root *must be* one or the other of two assigned expressions.

244. *In a quadratic equation where the terms are all on one side, and the coefficient of the square of the unknown quantity is unity, the sum of the roots is equal to the coefficient of the second term with its sign changed, and the product of the roots is equal to the last term.*

For let the equation be $x^2+px+q=0$;

the *sum* of the roots is

$$\frac{-p+\sqrt{(p^2-4q)}}{2}+\frac{-p-\sqrt{(p^2-4q)}}{2}, \text{ that is, } -p;$$

the product of the roots is

$$\frac{-p+\sqrt{(p^2-4q)}}{2}\times\frac{-p-\sqrt{(p^2-4q)}}{2},$$

that is $$\frac{p^2-(p^2-4q)}{4}, \text{ that is } q.$$

245. The preceding Article deserves special attention, for it furnishes a very good example both of the nature of the general results of Algebra, and of the methods by which these general results are obtained. The student should verify these results in the case of the quadratic equations already solved. Take, for example, that in Art. 232; the equation may be put in the form

$$x^2-\frac{4x}{3}-\frac{55}{3}=0,$$

and the roots are 5 and $-\frac{11}{3}$; thus the sum of the roots is $\frac{4}{3}$, and the product of the roots is $-\frac{55}{3}$.

246. Solve $$ax^2+bx+c=0.$$

By transposition, $$ax^2+bx=-c;$$

divide by a, $$x^2+\frac{bx}{a}=-\frac{c}{a};$$

add $\left(\frac{b}{2a}\right)^2$, $$x^2+\frac{bx}{a}+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2};$$

extract the square root, $$x+\frac{b}{2a}=\pm\frac{\sqrt{(b^2-4ac)}}{2a};$$

therefore $$x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}.$$

247. The general formulæ given in Arts. 241 and 246 may be employed in solving any quadratic equation. Take for example the equation $3x^2-4x-55=0$; divide by 3, thus we have

$$x^2-\frac{4x}{3}-\frac{55}{3}=0.$$

Take the formula in Art. 241, which gives the roots of $x^2+px+q=0$; and put $p=-\frac{4}{3}$, and $q=-\frac{55}{3}$; we shall thus obtain the roots of the proposed equation.

But it is more convenient to use the formula in Art. 246, as we thus avoid fractions. The proposed equation being $3x^2-4x-55=0$, we must put $a=3$, $b=-4$, and $c=-55$, in the formula which gives the roots of $ax^2+bx+c=0$,

that is, in $$\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}.$$

Thus we have $\frac{4\pm\sqrt{(16+660)}}{6}$, that is, $\frac{4\pm\sqrt{(676)}}{6}$,

that is, $\frac{4\pm 26}{6}$, that is, 5 or $-\frac{11}{3}$.

Examples. XXVI.

1. $2(x^2-7)+3(x^2-11)=33.$

2. $(x-15)(x+15)=400.$

3. $\dfrac{x^2-24}{5}+\dfrac{x^2-37}{4}=8.$

4. $\dfrac{3(x^2-11)}{5}-\dfrac{2(x^2-60)}{7}=36.$

5. $\dfrac{4}{x-3}-\dfrac{4}{x+3}=\dfrac{1}{3}.$

6. $\dfrac{x}{4}+\dfrac{4}{x}=\dfrac{x}{9}+\dfrac{9}{x}.$

7. $x^2-3x+2=0.$

8. $x^2-5x+6=0.$

9. $x^2+10x=24.$

10. $2x^2-1=5x+2.$

11. $3x^2-4x=39.$

12. $x^2+10x+3=2x^2-5x+53.$

13. $(x+1)(2x+3)=4x^2-22.$

14. $(x-1)(x-2)=20.$

15. $4(x^2-1)=4x-1.$

16. $(2x-3)^2=8x.$

17. $3x^2-17x+10=0.$

18. $\dfrac{9}{x}-\dfrac{x}{3}=2.$

19. $x=2+\dfrac{5}{4x}.$

20. $x^2-3=\dfrac{x-3}{6}.$

21. $\dfrac{2+x^2}{3}-\dfrac{x-x^2}{2}=1-x+x^2.$

22. $x+\dfrac{1}{x-3}=5.$

23. $4x-\dfrac{12-x}{x-3}=22.$

24. $\dfrac{2x+11}{x}=5-\dfrac{x-5}{3}.$

25. $\dfrac{x-1}{x-3}+2x=12.$

26. $\dfrac{x}{7}+\dfrac{21}{x+5}=6\frac{5}{7}.$

27. $8x+11+\dfrac{7}{x}=\dfrac{68x}{7}.$

28. $\dfrac{x+2}{x-2}+\dfrac{x-2}{x+2}=\dfrac{13}{6}.$

29. $\dfrac{2}{x+3}+\dfrac{x+3}{2}=\dfrac{10}{3}.$

30. $\dfrac{3(x-1)}{x+1}-\dfrac{2(x+1)}{x-1}=5.$

31. $\dfrac{2x}{x+2}+\dfrac{x+2}{2x}=2.$

32. $\dfrac{x}{x+1}+\dfrac{x+1}{x}=\dfrac{13}{6}.$

33. $\dfrac{x}{x+1}+\dfrac{x}{x+4}=1.$

34. $\dfrac{x+2}{x+1}+\dfrac{x+1}{x+2}=\dfrac{13}{6}.$

35. $\frac{x+1}{x-1}-\frac{x-2}{x+2}=\frac{9}{5}$.

36. $\frac{x+4}{x-4}+\frac{x+2}{x-2}=7$.

37. $\frac{x-2}{x-3}-\frac{x-4}{x-1}=\frac{14}{15}$.

38. $\frac{x-3}{x-2}-\frac{x-1}{x-4}=-\frac{6}{5}$.

39. $\frac{x-1}{x-4}-\frac{x-3}{x-2}=\frac{11}{12}$.

40. $\frac{1}{x-2}-\frac{2}{x+2}=\frac{3}{5}$.

41. $\frac{3}{2(x^2-1)}-\frac{1}{4(x+1)}=\frac{1}{8}$.

42. $\frac{x}{x^2-1}=\frac{15-7x}{8(1-x)}$.

43. $\frac{2x+1}{x-1}+\frac{3x-2}{3x+2}=\frac{11}{2}$.

44. $\frac{2x-1}{x-1}-\frac{2x-3}{x-2}+\frac{1}{6}=0$.

45. $\frac{3x+1}{3(x-5)}-\frac{2x-7}{2x-8}-\frac{5}{2}=0$.

46. $\frac{2x-3}{3x-5}+\frac{3x-5}{2x-3}=\frac{5}{2}$.

47. $\frac{3x-2}{2x-5}+\frac{2x-5}{3x-2}=\frac{10}{3}$.

48. $\frac{x+2}{x-1}-\frac{4-x}{2x}=\frac{7}{3}$.

49. $(x-3)^2=2(x^2-9)$.

50. $(x+10)^2=144(100-x^2)$.

51. $\frac{5}{x+2}+\frac{3}{x}=\frac{14}{x+4}$.

52. $\frac{4}{x+1}+\frac{5}{x+2}=\frac{12}{x+3}$.

53. $\frac{x+1}{x+2}+\frac{x-1}{x-2}=\frac{2x-1}{x-1}$.

54. $\frac{x-2}{x+2}+\frac{x+2}{x-2}=2\frac{x+3}{x-3}$.

55. $\frac{x-1}{x+1}-\frac{5}{6}=\frac{2}{7(x-1)}$.

56. $\frac{4}{x+2}+\frac{5}{x+4}=\frac{12}{x+6}$.

57. $\frac{x-1}{x+1}+\frac{x-2}{x+2}=\frac{2x+13}{x+16}$.

58. $\frac{x+1}{x-1}+\frac{x+2}{x-2}=\frac{2x+13}{x+1}$.

59. $\frac{2x-1}{x+1}+\frac{3x-1}{x+2}=\frac{5x-11}{x-1}$.

60. $x-\frac{14x-9}{8x-3}=\frac{x^2-3}{x+1}$.

61. $a^2x^2-2a^3x+a^4-1=0$.

62. $4a^2x=(a^2-b^2+x)^2$.

63. $\frac{x}{a}+\frac{a}{x}=\frac{x}{b}+\frac{b}{x}$.

64. $\frac{1}{x}+\frac{1}{x+b}=\frac{1}{a}+\frac{1}{a+b}$.

XXVII. *Equations which may be solved like Quadratics.*

248. There are many equations which are not strictly quadratics, but which may be solved by the method of *completing the square;* we will give two examples.

249. Solve $x^6 - 7x^3 = 8$.

Add $\left(\frac{7}{2}\right)^2$, $\quad x^6 - 7x^3 + \left(\frac{7}{2}\right)^2 = 8 + \frac{49}{4} = \frac{81}{4}$;

extract the square root, $x^3 - \frac{7}{2} = \pm\frac{9}{2}$;

therefore $\quad x^3 = \frac{7}{2} \pm \frac{9}{2} = 8$ or -1;

extract the cube root, thus $x = 2$ or -1.

250. Solve $x^2 + 3x + 3\sqrt{(x^2 + 3x - 2)} = 6$.

Subtract 2 from both sides, thus

$$x^2 + 3x - 2 + 3\sqrt{(x^2 + 3x - 2)} = 4.$$

Thus on the left-hand side we have two expressions, namely, $\sqrt{(x^2 + 3x - 2)}$ and $x^2 + 3x - 2$, and the latter is the square of the former; we can now *complete the square.*

Add $\left(\frac{3}{2}\right)^2$, thus

$$x^2 + 3x - 2 + 3\sqrt{(x^2 + 3x - 2)} + \left(\frac{3}{2}\right)^2 = 4 + \frac{9}{4} = \frac{25}{4};$$

extract the square root, thus

$$\sqrt{(x^2 + 3x - 2)} + \frac{3}{2} = \pm\frac{5}{2};$$

therefore $\quad \sqrt{(x^2 + 3x - 2)} = -\frac{3}{2} \pm \frac{5}{2} = 1$ or -4.

First suppose $\sqrt{(x^2+3x-2)}=1.$

Square both sides, thus $x^2+3x-2=1.$

This is an ordinary quadratic equation; by solving it we shall obtain $x=\frac{-3\pm\sqrt{21}}{2}$.

Next suppose $\sqrt{(x^2+3x-2)}=-4.$

Square both sides, thus $x^2+3x-2=16.$

This is an ordinary quadratic equation; by solving it we shall obtain $x=3$ or -6.

Thus on the whole we have four values for x, namely, 3 or -6 or $\frac{-3\pm\sqrt{21}}{2}$.

An important observation must be made with respect to these values. Suppose we proceed to verify them. If we put $x=3$ we find that $x^2+3x-2=16$, and thus $\sqrt{(x^2+3x-2)}=\pm 4$. If we take the value $+4$ the original equation will not be satisfied; if we take the value -4 it will be satisfied. If we put $x=-6$ we arrive at the same result. And the result might have been anticipated, because the values $x=3$ or -6 were obtained from $\sqrt{(x^2+3x-2)}=-4$, which was deduced from the original equation. If we put $x=\frac{-3\pm\sqrt{21}}{2}$ we find that $x^2+3x-2=1$, and the original equation will be satisfied if we take $\sqrt{(x^2+3x-2)}=+1$; and, as before, the result might have been anticipated.

In fact we shall find that we arrive at the same four values of x, by solving either of the following equations,

$$x^2+3x-3\sqrt{(x^2+3x-2)}=6,$$

$$x^2+3x+3\sqrt{(x^2+3x-2)}=6;$$

but the values 3 or -6 belong strictly only to the first equation, and the values $\frac{-3\pm\sqrt{21}}{2}$ belong strictly only to the second equation.

251. Equations may be proposed which will require the operations of transposing and squaring to be performed, once or oftener, before they are reduced to quadratics; we will give two examples.

252. Solve $2x - \sqrt{(x^2 - 3x - 3)} = 9.$

Transpose, $2x - 9 = \sqrt{(x^2 - 3x - 3)}$;

square, $4x^2 - 36x + 81 = x^2 - 3x - 3$;

transpose, $3x^2 - 33x + 84 = 0$;

divide by 3, $x^2 - 11x + 28 = 0.$

By solving this quadratic we shall obtain $x = 7$ or 4. The value 7 satisfies the original equation; the value 4 belongs strictly to the equation $2x + \sqrt{(x^2 - 3x - 3)} = 9.$

253. Solve $\sqrt{(x+4)} + \sqrt{(2x+6)} = \sqrt{(8x+9)}.$

Square, $x + 4 + 2x + 6 + 2\sqrt{(x+4)}\sqrt{(2x+6)} = 8x + 9$;

transpose, $2\sqrt{(x+4)}\sqrt{(2x+6)} = 5x - 1$;

square, $4(x+4)(2x+6) = 25x^2 - 10x + 1$;

that is, $8x^2 + 56x + 96 = 25x^2 - 10x + 1$;

transpose, $17x^2 - 66x - 95 = 0.$

By solving this quadratic we shall obtain $x = 5$ or $-\frac{19}{17}$. The value 5 satisfies the original equation; the value $-\frac{19}{17}$ belongs strictly to the equation

$$\sqrt{(2x+6)} - \sqrt{(x+4)} = \sqrt{(8x+9)}.$$

254. The student will see from the preceding examples that in cases in which we have to square in order to reduce an equation to the ordinary form, we cannot be certain without trial that the values finally obtained for the unknown quantity belong strictly to the original equation.

255. Equations are sometimes proposed which are intended to be solved, partly by inspection, and partly by ordinary methods; we will give two examples.

256. Solve $\frac{x+4}{x-4} - \frac{x-4}{x+4} = \frac{9+x}{9-x} - \frac{9-x}{9+x}$.

Bring the fractions on each side of the equation to a common denominator; thus

$$\frac{(x+4)^2-(x-4)^2}{x^2-16} = \frac{(9+x)^2-(9-x)^2}{81-x^2},$$

that is, $$\frac{16x}{x^2-16} = \frac{36x}{81-x^2}.$$

Here it is obvious that $x=0$ is a root. To find the other roots we begin by dividing both sides of the equation by $4x$; thus

$$\frac{4}{x^2-16} = \frac{9}{81-x^2};$$

therefore $$4(81-x^2) = 9(x^2-16);$$

therefore $$13x^2 = 324+144 = 468;$$

therefore $$x^2 = 36;$$

therefore $$x = \pm 6.$$

Thus there are three roots of the proposed equation, namely, 0, 6, -6.

257. Solve $$x^3 - 7xa^2 + 6a^3 = 0.$$

Here it is obvious that $x=a$ is a root. We may write the equation $x^3-a^3 = 7a^2(x-a)$; and to find the other roots we begin by dividing by $x-a$. Thus

$$x^2 + ax + a^2 = 7a^2.$$

By solving this quadratic we shall obtain $x=2a$ or $-3a$. Thus there are three roots of the proposed equation, namely, a, $2a$, $-3a$.

EXAMPLES. XXVII.

1. $x^4-13x^2+36=0.$ 2. $x-5\sqrt{x}-14=0.$

3. $x+\sqrt{(x+5)}=7.$ 4. $x^2+\sqrt{(x^2+9)}=21.$

5. $2\sqrt{(x^2-2x+1)}+x^2=23+2x.$

6. $x^4-2x^3+x^2=36.$ 7. $\sqrt{(x^2-6x+16)}+(x-3)^2=13.$

8. $9\sqrt{(x^2-9x+28)}+9x=x^2+36.$

9. $2x^2+6x=226-\sqrt{(x^2+3x-8)}.$

10. $x^4-4x^2-2\sqrt{(x^4-4x^2+4)}=31.$

11. $x+2\sqrt{(x^2+5x+2)}=10.$

12. $3x+\sqrt{(x^2+7x+5)}=19.$ 13. $x=7\sqrt{(2-x^2)}.$

14. $\sqrt{(x+9)}=2\sqrt{x}-3.$ 15. $\sqrt{(x+8)}-\sqrt{(x+3)}=\sqrt{x}.$

16. $5\sqrt{(1-x^2)}+5x=7.$

17. $\sqrt{(3x-3)}+\sqrt{(5x-19)}=\sqrt{(2x+8)}.$

18. $\sqrt{(2x+1)}+\sqrt{(7x-27)}=\sqrt{(3x+4)}.$

19. $\sqrt{(b^2+ax)}-\sqrt{(a^2+bx)}=a+b.$

20. $2x\sqrt{(a+x^2)}+2x^2=a^2-a.$

21. $\dfrac{x+\sqrt{(12a^2-x)}}{x-\sqrt{(12a^2-x)}}=\dfrac{a+1}{a-1}.$ 22. $\dfrac{1}{1-x}-\dfrac{1}{1+x}=\dfrac{3x}{1+x^2}.$

23. $\dfrac{1}{x+7}+\dfrac{1}{x-1}+\dfrac{1}{x+1}+\dfrac{1}{x-7}=0.$

24. $\dfrac{1}{x+\sqrt{(2-x^2)}}+\dfrac{1}{x-\sqrt{(2-x^2)}}=x.$

25. $\dfrac{x+\sqrt{(x^2-1)}}{x-\sqrt{(x^2-1)}}-\dfrac{x-\sqrt{(x^2-1)}}{x+\sqrt{(x^2-1)}}=8\sqrt{(x^2-1)}.$

26. $\dfrac{x+a}{x-a}-\dfrac{x-a}{x+a}=\dfrac{b+x}{b-x}-\dfrac{b-x}{b+x}.$

27. $x^3+3ax^2=4a^3.$ 28. $5x^2(a-x)=(a^2-x^2)(x+3a).$

XXVIII. *Problems which lead to Quadratic Equations.*

258. Find two numbers such that their sum is 15, and their product is 54.

Let x denote one of the numbers, then $15-x$ will denote the other number; and by supposition

$$x(15-x)=54.$$

By transposition, $$x^2-15x=-54;$$

therefore $$x^2-15x+\left(\frac{15}{2}\right)^2=-54+\frac{225}{4}=\frac{9}{4};$$

therefore $$x-\frac{15}{2}=\pm\frac{3}{2};$$

therefore $$x=\frac{15}{2}\pm\frac{3}{2}=9 \text{ or } 6.$$

If we take $x=9$ we have $15-x=6$, and if we take $x=6$ we have $15-x=9$. Thus the two numbers are 6 and 9. Here although the quadratic equation gives two values of x, yet there is really only one solution of the problem.

259. A person laid out a certain sum of money in goods, which he sold again for £24, and lost as much per cent. as he laid out: find how much he laid out.

Let x denote the number of pounds which he laid out; then $x-24$ will denote the number of pounds which he lost. Now by supposition he lost at the rate of x per cent., that is the loss was the fraction $\frac{x}{100}$ of the cost; therefore

$$x\times\frac{x}{100}=x-24;$$

therefore $$x^2-100x=-2400.$$

From this quadratic equation we shall obtain $x=40$ or 60. Thus all we can infer is that the sum of money laid out was either £40 or £60; for each of these numbers satisfies all the conditions of the problem.

260. The sum of £7. 4*s*. was divided equally among a certain number of persons; if there had been two fewer persons, each would have received one shilling more: find the number of persons.

Let x denote the number of persons; then each person received $\frac{144}{x}$ shillings. If there had been $x-2$ persons each would have received $\frac{144}{x-2}$ shillings. Therefore, by supposition,

$$\frac{144}{x-2} = \frac{144}{x} + 1.$$

Therefore $$144x = 144(x-2) + x(x-2);$$

therefore $$x^2 - 2x = 288.$$

From this quadratic equation we shall obtain $x = 18$ or -16. Thus the number of persons must be 18, for that is the only number which satisfies the conditions of the problem. The student will naturally ask whether any meaning can be given to the other result, namely -16, and in order to answer this question we shall take another problem closely connected with that which we have here solved.

261. The sum of £7. 4*s*. was divided equally among a certain number of persons; if there had been two *more* persons, each would have received one shilling *less*: find the number of persons.

Let x denote the number of persons. Then proceeding as before we shall obtain the equation

$$\frac{144}{x+2} = \frac{144}{x} - 1;$$

therefore $$x^2 + 2x = 288;$$

therefore $$x = 16 \text{ or } -18.$$

Thus in the former problem we obtained an applicable result, namely 18, and an inapplicable result, namely -16; and in the present problem we obtain an applicable result, namely 16, and an inapplicable result, namely -18.

262. In solving problems it is often found, as in Art. 260, that results are obtained which do not apply to the problem actually proposed. The reason appears to be, that the algebraical mode of expression is more general than ordinary language, and thus the equation which is a proper representation of the conditions of the problem will also apply to other conditions. Experience will convince the student that he will always be able to select the result which belongs to the problem he is solving. And it will be often possible, by suitable changes in the enunciation of the original problem, to form a new problem corresponding to any result which was inapplicable to the original problem; this is illustrated in Article 261, and we will now give another example.

263. Find the price of eggs per score, when ten more in half a crown's worth lowers the price threepence per score.

Let x denote the number of pence in the price of a score of eggs, then each egg costs $\frac{x}{20}$ pence; and therefore the number of eggs which can be bought for half a crown is $30 \div \frac{x}{20}$, that is $\frac{600}{x}$. If the price were threepence per score less, each egg would cost $\frac{x-3}{20}$ pence, and the number of eggs which could be bought for half a crown would be $\frac{600}{x-3}$. Therefore, by supposition,

$$\frac{600}{x-3} = \frac{600}{x} + 10;$$

therefore $$60x = 60(x-3) + x(x-3);$$

therefore $$x^2 - 3x = 180.$$

From this quadratic equation we shall obtain $x = 15$ or -12. Hence the price required is 15*d.* per score. It will be found that 12*d.* is the result of the following problem; find the price of eggs per score when ten *fewer* in half a crown's worth *raises* the price threepence per score.

EXAMPLES. XXVIII.

1. Divide the number 60 into two parts such that their product may be 864.

2. The sum of two numbers is 60, and the sum of their squares is 1872: find the numbers.

3. The difference of two numbers is 6, and their product is 720: find the numbers.

4. Find three numbers such that the second shall be two-thirds of the first, and the third half of the first; and that the sum of the squares of the numbers shall be 549.

5. The difference of two numbers is 2, and the sum of their squares is 244: find the numbers.

6. Divide the number 10 into two parts such that their product added to the sum of their squares may make 76.

7. Find the number which added to its square root will make 210.

8. One number is 16 times another; and the product of the numbers is 144: find the numbers.

9. One hundred and ten bushels of coals were divided among a certain number of poor persons; if each person had received one bushel more he would have received as many bushels as there were persons: find the number of persons.

10. A company dining together at an inn find their bill amounts to £8. 15*s.*; two of them were not allowed to pay, and the rest found that their shares amounted to 10 shillings a man more than if all had paid: find the number of men in the company.

11. A cistern can be supplied with water by two pipes; by one of them it would be filled 6 hours sooner than by the other, and by both together in 4 hours: find the time in which each pipe alone would fill it.

12. A person bought a certain number of pieces of cloth for £33. 15*s*., which he sold again at £2. 8*s*. per piece, and he gained as much in the whole as a single piece cost: find the number of pieces of cloth.

13. *A* and *B* together can perform a piece of work in $14\frac{2}{5}$ days; and *A* alone can perform it in 12 days less than *B* alone: find the time in which *A* alone can perform it.

14. A man bought a certain quantity of meat for 18 shillings. If meat were to rise in price one penny per lb., he would get 3 lbs. less for the same sum. Find how much meat he bought.

15. The price of one kind of sugar per stone of 14 lbs. is 1*s*. 9*d*. more than that of another kind; and 8 lbs. less of the first kind can be got for £1 than of the second: find the price of each kind per stone.

16. A person spent a certain sum of money in goods, which he sold again for £24, and gained as much per cent. as the goods cost him: find what the goods cost.

17. The side of a square is 110 inches long: find the length and breadth of a rectangle which shall have its perimeter 4 inches longer than that of the square, and its area 4 square inches less than that of the square.

18. Find the price of eggs per dozen, when two less in a shilling's worth raises the price one penny per dozen.

19. Two messengers *A* and *B* were despatched at the same time to a place at the distance of 90 miles; the former by riding one mile per hour more than the latter arrived at the end of his journey one hour before him: find at what rate per hour each travelled.

20. A person rents a certain number of acres of pasture land for £70; he keeps 8 acres in his own possession, and sublets the remainder at 5 shillings per acre more than he gave, and thus he covers his rent and has £2 over: find the number of acres.

21. From two places at a distance of 320 miles, two persons A and B set out in order to meet each other. A travelled 8 miles a day more than B; and the number of days in which they met was equal to half the number of miles B went in a day. Find how far each travelled before they met.

22. A person drew a quantity of wine from a full vessel which held 81 gallons, and then filled up the vessel with water. He then drew from the mixture as much as he before drew of pure wine; and it was found that 64 gallons of pure wine remained. Find how much he drew each time.

23. A certain company of soldiers can be formed into a solid square; a battalion consisting of seven such equal companies can be formed into a hollow square, the men being four deep. The hollow square formed by the battalion is sixteen times as large as the solid square formed by one company. Find the number of men in the company.

24. There are three equal vessels A, B, and C; the first contains water, the second brandy, and the third brandy and water. If the contents of B and C be put together, it is found that the fraction obtained by dividing the quantity of brandy by the quantity of water is nine times as great as if the contents of A and C had been treated in like manner. Find the proportion of brandy to water in the vessel C.

25. A person lends £5000 at a certain rate of interest; at the end of a year he receives his interest, spends £25 of it, and adds the remainder to his capital; he then lends his capital at the same rate of interest as before, and at the end of another year finds that he has altogether £5382: determine the rate of interest.

XXIX. *Simultaneous Equations involving Quadratics.*

264. We shall now solve some examples of simultaneous equations involving quadratics. There are two cases of frequent occurrence for which rules can be given; in both these cases there are two unknown quantities and two equations. The unknown quantities will always be denoted by the letters x and y.

265. *First Case.* Suppose that one of the equations is of the first degree, and the other of the second degree.

Rule. *From the equation of the first degree find the value of either of the unknown quantities in terms of the other, and substitute this value in the equation of the second degree.*

Example. Solve $3x+4y=18$, $5x^2-3xy=2$.

From the first equation $y=\frac{18-3x}{4}$; substitute this value in the second equation; therefore

$$5x^2-\frac{3x(18-3x)}{4}=2;$$

therefore $$20x^2-54x+9x^2=8;$$

therefore $$29x^2-54x=8.$$

From this quadratic equation we find $x=2$ or $-\frac{4}{29}$; then by substituting in the value of y we find $y=3$ or $\frac{267}{58}$.

266. Solve $3x^2+5x-8y=36$, $2x^2-3x-4y=3$.

Here although neither of the given equations is of the first degree, yet we can immediately deduce from them an equation of the first degree.

For multiply the first equation by 2, and the second by 3; thus

$$6x^2+10x-16y=72, \quad 6x^2-9x-12y=9;$$

therefore, by subtraction, $10x-16y+9x+12y=72-9$;

that is, $$19x-4y=63.$$

From this equation we obtain $y=\frac{19x-63}{4}$; substitute this value in the first of the given equations; thus

$$3x^2+5x-2(19x-63)=36;$$

therefore $$3x^2-33x+90=0;$$

therefore $$x^2-11x+30=0.$$

From this quadratic equation we shall find that $x=5$ or 6; and then by substituting in the value of y we find that $y=8$ or $12\frac{3}{4}$.

267. *Second Case.* When the terms involving the unknown quantities in each equation constitute an expression which is homogeneous and of the second degree; see Art. 23.

Rule. *Assume* $y=vx$, *and substitute in both equations; then by division the value of* v *can be found.*

Example. Solve $x^2+xy+2y^2=44$, $2x^2-xy+y^2=16$.

Assume $y=vx$, and substitute for y; thus

$$x^2(1+v+2v^2)=44, \quad x^2(2-v+v^2)=16.$$

Therefore, by division,

$$\frac{1+v+2v^2}{2-v+v^2}=\frac{44}{16}=\frac{11}{4};$$

therefore $$4(1+v+2v^2)=11(2-v+v^2);$$

therefore $$3v^2-15v+18=0;$$

therefore $$v^2-5v+6=0.$$

From this quadratic equation we shall obtain $v=2$ or 3. In the equation $x^2(1+v+2v^2)=44$ put 2 for v; thus $x=\pm 2$; and since $y=vx$, we have $y=\pm 4$. Again, in the same equation put 3 for v; thus $x=\pm\sqrt{2}$; and since $y=vx$, we have $y=\pm 3\sqrt{2}$.

Or we might proceed thus: multiply the first of the given equations by 2; thus

$$2x^2+2xy+4y^2=88;$$

the second equation is $2x^2-xy+y^2=16$.

By subtraction $3xy+3y^2=72$, therefore $y^2=24-xy$.

Again, multiply the second equation by 2 and subtract the first equation; thus

$$3x^2-3xy=-12; \text{ therefore } x^2=xy-4.$$

Hence, by multiplication

$$x^2y^2=(24-xy)(xy-4),$$

or $$2x^2y^2-28xy=-96.$$

By solving this quadratic we obtain $xy=8$ or 6. Substitute the former in the given equations; thus

$$x^2+2y^2=36, \quad 2x^2+y^2=24.$$

Hence we can find x^2 and y^2. Similarly we may take the other value of xy, and then find x^2 and y^2.

268. Solve $2x^2+3xy+y^2=70$, $6x^2+xy-y^2=50$.

Assume $y=vx$, and substitute for y; thus

$$x^2(2+3v+v^2)=70, \quad x^2(6+v-v^2)=50.$$

Therefore by division

$$\frac{2+3v+v^2}{6+v-v^2}=\frac{70}{50}=\frac{7}{5};$$

therefore $$5(2+3v+v^2)=7(6+v-v^2);$$

therefore $$12v^2+8v-32=0;$$

therefore $$3v^2+2v-8=0.$$

From this quadratic equation we shall find $v=\frac{4}{3}$ or -2.

In the equation $x^2(2+3v+v^2)=70$ put $\frac{4}{3}$ for v; thus $x=\pm 3$; and since $y=vx$ we have $y=\pm 4$. The value $v=-2$ we shall find to be inapplicable; for it leads to the inadmissible result $x^2\times 0=70$. In fact the equations from which the value of v was obtained may be written thus,

$$x^2(2+v)(1+v)=70, \qquad x^2(2+v)(3-v)=50;$$

and hence we see that the value of v found from $2+v=0$ is inapplicable, and that we can only have

$$\frac{1+v}{3-v}=\frac{70}{50}=\frac{7}{5}; \text{ and therefore } v=\frac{4}{3}.$$

269. Equations may be proposed which do not fall under either of the two cases which we have discussed, but which may be solved by artifices which can only be suggested by trial and experience. We will give some examples.

270. Solve $\qquad x+y=5, \qquad x^3+y^3=65.$

By division, $$\frac{x^3+y^3}{x+y}=\frac{65}{5},$$

that is, $$x^2-xy+y^2=13;$$

then from this equation combined with $x+y=5$ we can find x and y by the first case. Or we may complete the solution thus,

$$x+y=5;$$

square $$x^2+2xy+y^2=25 \text{ .... (1).}$$

Also $$x^2-xy+y^2=13 \text{ .... (2).}$$

Therefore, by subtraction, $\qquad 3xy=12;$

therefore $$xy=4;$$

therefore $$4xy=16 \text{ .... (3).}$$

Subtract (3) from (1); thus

$$x^2-2xy+y^2=9;$$

extract the square root, $\qquad x-y=\pm 3.$

We have now to find x and y from the simple equations

$$x+y=5, \qquad x-y=\pm 3;$$

these lead to $\quad x=1$ or $4, \quad y=4$ or 1.

271. Solve $\quad x^2+y^2=41, \qquad xy=20.$

These equations can be solved by the second case; or they may be solved in the manner just exemplified. For we can deduce from them

$$x^2+y^2+2xy=41+40=81,$$
$$x^2+y^2-2xy=41-40=1;$$

then by extracting the square roots,

$$x+y=\pm 9, \qquad x-y=\pm 1.$$

And thus finally we shall obtain

$$x=\pm 5 \text{ or } \pm 4, \qquad y=\pm 4 \text{ or } \pm 5.$$

272. Solve $\quad x^2+xy+y^2=19, \quad x^4+x^2y^2+y^4=133.$

By division, $\quad \dfrac{x^4+x^2y^2+y^4}{x^2+xy+y^2}=\dfrac{133}{19};$

that is, $\quad x^2-xy+y^2=7.$

We have now to solve the equations

$$x^2+xy+y^2=19, \qquad x^2-xy+y^2=7.$$

By addition and subtraction we obtain successively

$$x^2+y^2=13, \qquad xy=6.$$

Then proceeding as in Art. 271, we shall find

$$x=\pm 3 \text{ or } \pm 2, \qquad y=\pm 2 \text{ or } \pm 3.$$

273. Solve $\quad x-y=2, \qquad x^5-y^5=242.$

By division, $\quad \dfrac{x^5-y^5}{x-y}=\dfrac{242}{2};$

that is, $\quad x^4+x^3y+x^2y^2+xy^3+y^4=121,$

that is, $\quad x^4+y^4+xy(x^2+y^2)+x^2y^2=121$ .... (1).

Now $\quad x-y=2;$

square $\quad x^2-2xy+y^2=4;$

therefore $\quad x^2+y^2=2xy+4$ .... (2).

Square $x^4+2x^2y^2+y^4=4x^2y^2+16xy+16$;

therefore $x^4+y^4=2x^2y^2+16xy+16$ .... (3).

Substitute from (2) and (3) in (1); thus

$$2x^2y^2+16xy+16+xy(2xy+4)+x^2y^2=121;$$

that is, $5x^2y^2+20xy=105$;

therefore $x^2y^2+4xy=21$.

From this quadratic equation we shall obtain $xy=3$ or -7. Take $xy=3$, and from this combined with $x-y=2$, we shall obtain $x=3$ or -1, $y=1$ or -3. If we take $xy=-7$, we shall find that the values of x and y are impossible; see Art. 236.

EXAMPLES. XXIX.

1. $x-y=1, \quad x^2-xy+y^2=21.$
2. $2x-5y=3, \quad x^2+xy=20.$
3. $x+y=7(x-y), \quad x^2+y^2=100.$
4. $5(x^2-y^2)=4(x^2+y^2), \quad x+y=8.$
5. $x-y=3, \quad x^2+y^2=65.$
6. $4x-5y=1, \quad 2x^2-xy+3y^2+3x-4y=47.$
7. $4x+9y=12, \quad 2x^2+xy=6y^2.$
8. $(x-6)^2+(y-5)^2+2xy=60, \quad 5y-4x=1.$
9. $4x^2+2xy+\frac{y^2}{4}+\frac{5}{12}(4x+y)=41, \quad 4x-y=4.$
10. $\frac{x}{12}+\frac{y}{10}=x-y, \quad \frac{7xy}{15}-\frac{2x}{3}-2y=0.$
11. $3x+2y=5xy, \quad 15x-4y=4xy.$
12. $xy+2=9y, \quad xy+2=x.$
13. $8(xy+1)=33y, \quad 4(xy+1)=33x.$
14. $xy=x+y, \quad ax=by.$

15. $\frac{x}{a}+\frac{y}{b}=2, \quad xy=ab.$

16. $\frac{x}{a}+\frac{y}{b}=2, \quad \frac{x^2}{a}+\frac{y^2}{b}=a+b.$

17. $\frac{x}{a}+\frac{y}{b}=2, \quad x^2+y^2=ax+by.$

18. $\frac{x}{a}+\frac{y}{b}=1, \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

19. $x^2+xy=28, \quad xy-y^2=3.$

20. $x^2+xy=45, \quad y^2+xy=36.$

21. $2x^2-xy=56, \quad 2xy-y^2=48.$

22. $x^2-2xy=15, \quad xy-2y^2=7.$

23. $x^2+3xy=28, \quad xy+4y^2=8.$

24. $x^2+xy-6y^2=21, \quad xy-2y^2=4.$

25. $x^2+3xy=54, \quad xy+4y^2=115.$

26. $\frac{x+y}{x-y}+\frac{x-y}{x+y}=\frac{5}{2}, \quad x^2+y^2=90.$

27. $\frac{x^2+y^2}{x^2-y^2}=\frac{25}{7}, \quad xy=48.$

28. $\frac{x+y}{x-y}+\frac{x-y}{x+y}=\frac{10}{3}, \quad x^2-y^2=3.$

29. $x(x+y)+y(x-y)=158, \quad 7x(x+y)=72y(x-y).$

30. $x^2y(x+y)=80, \quad x^2y(2x-3y)=80.$

31. $2x^2-xy+y^2=2y, \quad 2x^2+4xy=5y.$

32. $\frac{x+y}{x-y}+\frac{x-y}{x+y}=\frac{a^2+1}{a}, \quad x^2+y^2=b^2.$

33. $x^2+xy=a(a+b), \quad x^2+y^2=a^2+b^2.$

34. $x^2+2xy-y^2=a^2+2a-1,$
$(a-1)x(x+y)=a(a+1)y(x-y).$

35. $x-y=2, \quad x^3-y^3=152.$

36. $x+y=9, \quad x^3+y^3=189.$

37. $x^2+y^2=20, \quad xy-x-y=2.$

38. $x-y=1, \quad x^5-y^5=781.$

39. $x+y=3, \quad x^5+y^5=33.$

40. $x^2+xy+y^2=37, \quad x^4+x^2y^2+y^4=481.$

41. $\dfrac{x}{x-y}-\dfrac{x-y}{x+y}=1, \quad 2+3xy=3x.$

42. $x^2+y^2=34, \quad x^2-y^2+\sqrt{(x^2-y^2)}=20.$

43. $x^2+y^2-1=2xy, \quad xy(xy+1)=6.$

44. $4x^2+y^2+2(2x+y)=6, \quad 4xy(xy+1)=3.$

45. $x^2+xy=8x+3, \quad y^2+xy=8y+6.$

46. $x^2-xy=2x+5, \quad xy-y^2=2y+2.$

47. $2x+y+6\sqrt{(2x+y+4)}=23, \quad 4x^2-6x=y^2+3y.$

48. $18+9(x+y)=2(x+y)^2, \quad 6-(x-y)=(x-y)^2.$

49. $x^2-xy=a(x+1)+b+1, \quad xy-y^2=ay+b.$

50. $\dfrac{a^2}{x^2}+\dfrac{y^2}{b^2}=18, \quad \dfrac{ab}{xy}=1.$

51. $\dfrac{a^2}{x^2}-\dfrac{y^2}{b^2}=12, \quad \dfrac{ab}{xy}=2.$

52. $x^2=ax+by, \quad y^2=ay+bx.$

53. $x^2yz=a, \quad xy^2z=b, \quad xyz^2=c.$

54. $(x+y)(x+z)=a^2, \ (y+z)(y+x)=b^2, \ (z+x)(z+y)=c^2.$

55. $3yz+2zx-4xy=16, \quad 2yz-3zx+xy=5,$
$4yz-zx-3xy=15.$

56. $6(x^2+y^2+z^2)=13(x+y+z)=\dfrac{481}{6}, \quad xy=z^2.$

XXX. *Problems which lead to Quadratic Equations with more than one unknown quantity.*

274. There is a certain number of two digits; the sum of the squares of the digits is equal to the number increased by the product of its digits; and if thirty-six be added to the number the digits are reversed: find the number.

Let x denote the digit in the tens' place, and y the digit in the units' place. Then the number is $10x+y$; and if the digits be reversed we obtain $10y+x$. Therefore, by supposition, we have

$$x^2+y^2-xy+10x+y \ldots\ldots\ldots\ldots(1).$$

$$10x+y+36=10y+x \ldots\ldots\ldots\ldots(2).$$

From (2) we obtain $9y=9x+36$; therefore $y=x+4$.

Substitute in (1), thus

$$x^2+(x+4)^2=x(x+4)+10x+x+4;$$

therefore $$x^2-7x+12=0.$$

From this quadratic equation we obtain $x=3$ or 4; and therefore $y=7$ or 8. Hence the required number must be either 37 or 48; each of these numbers satisfies all the conditions of the problem.

275. A man starts from the foot of a mountain to walk to its summit. His rate of walking during the second half of the distance is half a mile per hour less than his rate during the first half, and he reaches the summit in $5\frac{1}{2}$ hours. He descends in $3\frac{3}{4}$ hours by walking at a uniform rate, which is one mile per hour more than his rate during the first half of the ascent. Find the distance to the summit, and his rates of walking.

Let $2x$ denote the number of miles to the summit, and suppose that during the first half of the ascent the man

walked y miles per hour. Then he took $\frac{x}{y}$ hours for the first half of the ascent, and $\frac{x}{y-\frac{1}{2}}$ hours for the second.

Therefore $\frac{x}{y}+\frac{x}{y-\frac{1}{2}}=5\frac{1}{2}$ .... (1).

Similarly, $\frac{2x}{y+1}=3\frac{3}{4}$ .... (2).

From (2), $2x=\frac{15}{4}(y+1)$;

therefore $x=\frac{15}{8}(y+1)$.

From (1), $x\left(2y-\frac{1}{2}\right)=\frac{11}{2}y\left(y-\frac{1}{2}\right)$.

Therefore, by substitution,

$$\frac{15}{8}(y+1)\left(2y-\frac{1}{2}\right)=\frac{11}{2}y\left(y-\frac{1}{2}\right);$$

therefore $15(y+1)(4y-1)=44y(2y-1)$;

therefore $28y^2-89y+15=0$.

From this quadratic equation we obtain $y=3$ or $\frac{5}{28}$. The value $\frac{5}{28}$ is inapplicable, because by supposition y is greater than $\frac{1}{2}$. Therefore $y=3$; and then $x=\frac{15}{2}$, so that the whole distance to the summit is 15 miles.

EXAMPLES. XXX.

1. The sum of the squares of two numbers is 170, and the difference of their squares is 72: find the numbers.

2. The product of two numbers is 108, and their sum is twice their difference: find the numbers.

3. The product of two numbers is 192, and the sum of their squares is 640: find the numbers.

4. The product of two numbers is 128, and the difference of their squares is 192: find the numbers.

5. The product of two numbers is 6 times their sum, and the sum of their squares is 325: find the numbers.

6. The product of two numbers is 60 times their difference, and the sum of their squares is 244: find the numbers.

7. The sum of two numbers is 6 times their difference, and their product exceeds their sum by 23: find the numbers.

8. Find two numbers such that twice the first with three times the second may make 60, and twice the square of the first with three times the square of the second may make 840.

9. Find two numbers such that their difference multiplied into the difference of their squares shall make 32, and their sum multiplied into the sum of their squares shall make 272.

10. Find two numbers such that their difference added to the difference of their squares may make 14, and their sum added to the sum of their squares may make 26.

11. Find two numbers such that their product is equal to their sum, and their sum added to the sum of their squares equal to 12.

12. Find two numbers such that their sum increased by their product is equal to 34, and the sum of their squares diminished by their sum equal to 42.

13. The difference of two numbers is 3, and the difference of their cubes is 279: find the numbers.

14. The sum of two numbers is 20, and the sum of their cubes is 2240: find the numbers.

15. A certain rectangle contains 300 square feet; a second rectangle is 8 feet shorter, and 10 feet broader, and also contains 300 square feet: find the length and breadth of the first rectangle.

16. A person bought two pieces of cloth of different sorts; the finer cost 4 shillings a yard more than the coarser, and he bought 10 yards more of the coarser than of the finer. For the finer piece he paid £18, and for the coarser piece £16. Find the number of yards in each piece.

17. A man has to travel a certain distance; and when he has travelled 40 miles he increases his speed 2 miles per hour. If he had travelled with his increased speed during the whole of his journey he would have arrived 40 minutes earlier; but if he had continued at his original speed he would have arrived 20 minutes later. Find the whole distance he had to travel, and his original speed.

18. A number consisting of two digits has one decimal place; the difference of the squares of the digits is 20, and if the digits be reversed, the sum of the two numbers is 11: find the number.

19. A person buys a quantity of wheat which he sells so as to gain 5 per cent. on his outlay, and thus clears £16. If he had sold it at a gain of 5 shillings per quarter, he would have cleared as many pounds as each quarter cost him shillings: find how many quarters he bought, and what each quarter cost.

20. Two workmen, A and B, were employed by the day at different rates; A at the end of a certain number of days received £4. 16*s*., but B, who was absent six of

those days, received only £2. 14*s*. If *B* had worked the whole time, and *A* had been absent six days, they would have received exactly alike. Find the number of days, and what each was paid per day.

21. Two trains start at the same time from two towns, and each proceeds at a uniform rate towards the other town. When they meet it is found that one train has run 108 miles more than the other, and that if they continue to run at the same rate they will finish the journey in 9 and 16 hours respectively. Find the distance between the towns and the rates of the trains.

22. *A* and *B* are two towns situated 18 miles apart on the same bank of a river. A man goes from *A* to *B* in 4 hours, by rowing the first half of the distance and walking the second half. In returning he walks the first half at the same rate as before, but the stream being with him, he rows $1\frac{1}{2}$ miles per hour more than in going, and accomplishes the whole distance in $3\frac{1}{2}$ hours. Find his rates of walking and rowing.

23. *A* and *B* run a race round a two mile course. In the first heat *B* reaches the winning post 2 minutes before *A*. In the second heat *A* increases his speed 2 miles per hour, and *B* diminishes his as much; and *A* then arrives at the winning post two minutes before *B*. Find at what rate each man ran in the first heat.

24. Two travellers, *A* and *B*, set out from two places, *P* and *Q*, at the same time; *A* starts from *P* with the design to pass through *Q*, and *B* starts from *Q* and travels in the same direction as *A*. When *A* overtook *B* it was found that they had together travelled thirty miles, that *A* had passed through *Q* four hours before, and that *B*, at his rate of travelling, was nine hours' journey distant from *P*. Find the distance between *P* and *Q*.

XXXI. *Involution.*

276. We have already defined a *power* to be the product of two or more *equal factors*, and we have explained the notation for denoting powers; see Arts. 15, 16, 17. The process of obtaining powers is called *Involution;* so that Involution is only a particular case of Multiplication, but it is a particular case which occurs so often that it is convenient to devote a Chapter to it. The student will find that he is already familiar with some of the results which we shall have to notice, and that the whole of the present Chapter follows immediately from the elementary laws of Algebra.

277. *Any even power of a negative quantity is positive, and any odd power is negative.*

This is a simple consequence of the *Rule of Signs*. Thus, for example, $-a \times -a = a^2$, $-a \times -a \times -a = a^2 \times -a = -a^3$; $-a \times -a \times -a \times -a = -a^3 \times -a = a^4$; and so on. In the following Articles, when we use the words *give the proper sign*, we mean that the sign is to be determined by the rule of the present Article. (See Art. 38.)

278. Rule for obtaining a power of a power. *Multiply the numbers denoting the powers for the new exponent, and give the proper sign to the result.*

Thus, for example, $(a^2)^3 = a^6$; $(-a^3)^3 = -a^9$; $(a^4)^3 = a^{12}$; $(-a^4)^3 = -a^{12}$. This is a simple consequence of the law of powers which is demonstrated in Art. 59. For example,

$$(a^2)^3 = a^2 \times a^2 \times a^2 = a^{2+2+2} = a^{2\times3} = a^6.$$

The Rule of the present Article leads immediately to that which we shall now give.

279. Rule for obtaining any power of a simple integral expression. *Multiply the index of every factor in the expression by the number denoting the power, and give the proper sign to the result.*

Thus, for example,

$$(a^2b^3)^2 = a^4b^6;\ (-a^2b^3)^3 = -a^6b^9;\ (ab^2c^3)^4 = a^4b^8c^{12};$$
$$(-a^2b^3c^4)^5 = -a^{10}b^{15}c^{20};\ (2ab^2c^3)^6 = 2^6a^6b^{12}c^{18} = 64a^6b^{12}c^{18}.$$

280. Rule for obtaining any power of a fraction. *Raise both the numerator and denominator to that power, and give the proper sign to the result.*

This follows from Art. 145. For example,

$$\left(\frac{a^2}{b^3}\right)^2 = \frac{a^4}{b^6};\quad \left(-\frac{a^2}{b^3}\right)^3 = -\frac{a^6}{b^9};\quad \left(\frac{2a^2}{3b}\right)^4 = \frac{2^4a^8}{3^4b^4} = \frac{16a^8}{81b^4}.$$

281. Some examples of Involution in the case of *binomial expressions* have already been given. See Arts. 82 and 88. Thus

$$(a+b)^2 = a^2 + 2ab + b^2,$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

The student may for exercise obtain the fourth, fifth and sixth powers of $a+b$. It will be found that

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4,$$
$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$
$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

In like manner the following results may be obtained:

$$(a-b)^2 = a^2 - 2ab + b^2,$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3,$$
$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4,$$
$$(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$$
$$(a-b)^6 = a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6.$$

Thus in the results obtained for the powers of $a-b$, where any *odd* power of b occurs, the negative sign is prefixed; and thus any power of $a-b$ can be immediately deduced from the same power of $a+b$, by changing the signs of the terms which involve the odd powers of b.

282. The student will see hereafter that, by the aid of a theorem called the *Binomial Theorem*, any power of a binomial expression can be obtained without the labour of actual multiplication.

283. The formulæ given in Article 281 may be used in the way we have already explained in Art. 84. Suppose, for example, we require the fourth power of $2x-3y$. In the formula for $(a-b)^4$ put $2x$ for a, and $3y$ for b; thus,

$$(2x-3y)^4=(2x)^4-4(2x)^3(3y)+6(2x)^2(3y)^2-4(2x)(3y)^3+(3y)^4$$

$$=16x^4-96x^3y+216x^2y^2-216xy^3+81y^4.$$

284. It will be easily seen that we can obtain required results in Involution by different processes. Suppose, for example, that we require the sixth power of $a+b$. We may obtain this by repeated multiplication by $a+b$. Or we may first find the cube of $a+b$, and then the square of this result; since the square of $(a+b)^3$ is $(a+b)^6$. Or we may first find the square of $a+b$, and then the cube of this result; since the cube of $(a+b)^2$ is $(a+b)^6$. In like manner the eighth power of $a+b$ may be found by taking the square of $(a+b)^4$, or by taking the fourth power of $(a+b)^2$.

285. Some examples of Involution in the case of *trinomial expressions* have already been given. See Arts. 85 and 88. Thus

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ac,$$

$$(a+b+c)^3=$$
$$a^3+b^3+c^3+3a^2(b+c)+3b^2(a+c)+3c^2(a+b)+6abc.$$

These formulæ may be used in the manner explained in Art. 84. Suppose, for example, we require $(1-2x+3x^2)^2$. In the formula for $(a+b+c)^2$ put 1 for a, $-2x$ for b, and $3x^2$ for c; thus we obtain

$$(1-2x+3x^2)^2=$$
$$(1)^2+(-2x)^2+(3x^2)^2+2(1)(-2x)+2(-2x)(3x^2)+2(1)(3x^2)$$
$$=1+4x^2+9x^4-4x-12x^3+6x^2$$
$$=1-4x+10x^2-12x^3+9x^4.$$

Similarly, we have

$$(1-2x+3x^2)^3 =$$
$$1^3+(-2x)^3+(3x^2)^3$$
$$+3(1)^2(-2x+3x^2)+3(-2x)^2(1+3x^2)+3(3x^2)^2(1-2x)$$
$$+6(1)(-2x)(3x^2)$$
$$=1-8x^3+27x^6$$
$$+3(-2x+3x^2)+12x^2(1+3x^2)+27x^4(1-2x)-36x^3$$
$$=1-6x+21x^2-44x^3+63x^4-54x^5+27x^6.$$

286. It is found by observation that the square of any multinomial expression may be obtained by either of two rules. Take, for example, $(a+b+c+d)^2$. It will be found that this

$$=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd;$$

and this may be obtained by the following rule; *the square of any multinomial expression consists of the square of each term, together with twice the product of every pair of terms.*

Again, we may put the result in this form

$$(a+b+c+d)^2$$
$$=a^2+2a(b+c+d)+b^2+2b(c+d)+c^2+2cd+d^2,$$

and this may be obtained by the following rule; *the square of any multinomial expression consists of the square of each term, together with twice the product of each term by the sum of all the terms which follow it.*

EXAMPLES. XXXI.

Find

1. $(2x^2y^3z^4)^3$.
2. $(-2x^2y^2z^3)^3$.
3. $(-3ab^2c^3)^4$.
4. $\left(\dfrac{2x^2}{3y^2}\right)^2$.

5. $\left(-\frac{4x}{3y^2}\right)^3$.

6. $\left(-\frac{x^3}{y^2z^2}\right)^4$.

7. $(a+b)^7$.

8. $(a-b)^7$.

9. $(a+b)^3(a-b)^3$.

10. $(1-x)^3$.

11. $(2+x)^3$.

12. $(3-2x)^3$.

13. $(1+x)^4$.

14. $(x-2)^4$.

15. $(2x+3)^4$.

16. $(ax+by)^3+(ax-by)^3$.

17. $(ax+by)^4+(ax-by)^4$.

18. $(1+x)^5-(1-x)^5$.

19. $(1+x)^4(1-x)^4$.

20. $(1+x+x^2)^2$.

21. $(1-x+x^2)^2$.

22. $(1+x-x^2)^2$.

23. $(1+3x+2x^2)^2$.

24. $(1-3x+3x^2)^2$.

25. $(2+3x+4x^2)^2+(2-3x+4x^2)^2$.

26. $(1+x+x^2)^3$.

27. $(1-x+x^2)^3$.

28. $(1+x-x^2)^3$.

29. $(1+3x+2x^2)^3$.

30. $(1-3x+3x^2)^3$.

31. $(2+3x+4x^2)^3-(2-3x+4x^2)^3$.

32. $(1-x+x^2+x^3)^2$.

33. $(1+2x+3x^2+4x^3)^2$.

34. $(a+b+c+d)^2-(a-b+c-d)^2$.

35. $(a+b+c+d)^2+(a-b+c-d)^2$.

36. $(1+3x+3x^2+x^3)^2$.

37. $(1-6x+12x^2-8x^3)^2$.

38. $(1+4x+6x^2+4x^3+x^4)^2$.

39. $(1-x)^3(1+x+x^2)^3$.

40. $(1-x+x^2)^3(1+x+x^2)^3$.

XXXII. *Evolution.*

287. Evolution is the inverse of Involution; so that Evolution is the method of finding any proposed root of a given number or expression. It is usual to employ the word *extract* and its derivatives in connexion with the word *root;* thus, for example, *to extract the square root* means the same thing as to *find the square root.*

In the present Chapter we shall begin by stating three simple consequences of the *Rule of Signs*, we shall then consider in succession the extraction of the roots of simple expressions, the extraction of the square root of compound expressions and numbers, and the extraction of the cube root of compound expressions and numbers.

288. *Any even root of a positive quantity may be either positive or negative.*

Thus, for example, $a \times a = a^2$, and $-a \times -a = a^2$; therefore the square root of a^2 is either a or $-a$, that is, either $+a$ or $-a$.

289. *Any odd root of a quantity has the same sign as the quantity.*

Thus, for example, the cube root of a^3 is a, and the cube root of $-a^3$ is $-a$.

290. *There can be no even root of a negative quantity.*

Thus, for example, there can be no square root of $-a^2$; for if any quantity be multiplied by itself the result is a *positive* quantity.

The fact that there can be no even root of a negative quantity is sometimes expressed by calling such a root an *impossible quantity* or an *imaginary quantity.*

291. Rule for obtaining any root of a simple integral expression. *Divide the index of every factor in the expression by the number denoting the root, and give the proper sign to the result.*

Thus, for example, $\sqrt{(16a^2b^4)} = \sqrt{(4^2a^2b^4)} = \pm 4ab^2$.

$\sqrt[3]{(-8a^6b^9c^{12})} = \sqrt[3]{(-2^3a^6b^9c^{12})} = -2a^2b^3c^4$.

$\sqrt[4]{(256x^4y^8)} = \sqrt{(4^4x^4y^8)} = \pm 4xy^2$.

292. Rule for obtaining any root of a fraction. *Find the root of the numerator and denominator, and give the proper sign to the result.*

For example, $\sqrt{\left(\frac{4a^2}{9b^4}\right)} = \sqrt{\left(\frac{2^2a^2}{3^2b^4}\right)} = \pm\frac{2a}{3b^2}$.

$$\sqrt[3]{\left(-\frac{27a^6}{64b^3}\right)} = \sqrt[3]{\left(-\frac{3^3a^6}{4^3b^3}\right)} = -\frac{3a^2}{4b}.$$

293. Suppose we require the cube root of a^2. In this case the index 2 is not divisible by the number 3 which denotes the required root; and we have, at present, no other mode of expressing the result than $\sqrt[3]{a^2}$. Similarly, $\sqrt{a}$, $\sqrt{a^3}$, $\sqrt[4]{a^5}$, cannot, at present, be otherwise expressed. Such quantities are called *surds* or *irrational* quantities; and we shall consider them in the next two Chapters.

294. We now proceed to the method of extracting the square root of a compound expression.

The square root of $a^2 + 2ab + b^2$ is $a + b$; and we shall be led to a general rule for the extraction of the square root of any compound expression by observing the manner in which $a + b$ may be derived from $a^2 + 2ab + b^2$.

$$\begin{array}{r|l} & a^2 + 2ab + b^2 \; (a + b \\ & a^2 \\ \hline 2a + b) & 2ab + b^2 \\ & 2ab + b^2 \\ \hline \end{array}$$

Arrange the terms according to the dimensions of one letter a; then the first term is a^2, and its square root is a, which is the first term of the required root. Subtract its square, that is a^2, from the whole expression, and bring down the remainder $2ab + b^2$. Divide $2ab$ by $2a$, and the quotient is b, which is the other term of the required root. Take twice the first term and add the second term, that is, take $2a + b$; multiply this by the second term, that is by b, and subtract the product, that is $2ab + b^2$, from the remainder. This finishes the operation in the present case.

If there were more terms we should proceed with $a+b$ as we did formerly with a; its square, that is, $a^2+2ab+b^2$, has already been subtracted from the proposed expression, so we should divide the remainder by $2(a+b)$ for a new term in the root. Then for a new subtrahend we multiply the sum of $2(a+b)$ and the new term, by the new term. The process must be continued until the required root is found.

295. Examples.

$$4x^2+12xy+9y^2\,(2x+3y$$
$$4x^2$$
$$4x+3y)\ 12xy+9y^2$$
$$12xy+9y^2$$

$$4x^4-20x^3+37x^2-30x+9\,(2x^2-5x+3$$
$$4x^4$$
$$4x^2-5x)\ -20x^3+37x^2-30x+9$$
$$-20x^3+25x^2$$
$$4x^2-10x+3)\ 12x^2-30x+9$$
$$12x^2-30x+9$$

$$x^4-4x^3y+10x^2y^2-12xy^3+9y^4\,(x^2-2xy+3y^2$$
$$x^4$$
$$2x^2-2xy)\ -4x^3y+10x^2y^2-12xy^3+9y^4$$
$$-4x^3y^2+\ 4x^2y^2$$
$$2x^2-4xy+3y^2)\ 6x^2y^2-12xy^3+9y^4$$
$$6x^2y^2-12xy^3+9y^4$$

$$
\begin{array}{rl}
 & x^6 + 4x^5 \quad - 10x^3 \quad + 4x + 1 \;(x^3 + 2x^2 - 2x - 1 \\
 & x^6 \\
\hline
2x^3 + 2x^2 \,) & 4x^5 \quad - 10x^3 \quad + 4x + 1 \\
 & 4x^5 + 4x^4 \\
\hline
2x^3 + 4x^2 - 2x \,) & -4x^4 - 10x^3 \quad + 4x + 1 \\
 & -4x^4 - 8x^3 + 4x^2 \\
\hline
2x^3 + 4x^2 - 4x - 1 \,) & -2x^3 - 4x^2 + 4x + 1 \\
 & -2x^3 - 4x^2 + 4x + 1 \\
\hline
\end{array}
$$

296. It has been already observed that all even roots admit of a double sign; see Art. 288. Thus the square root of $a^2 + 2ab + b^2$ is either $a + b$ or $-a - b$. In fact, in the process of extracting the square root of $a^2 + 2ab + b^2$, we begin by extracting the square root of a^2; and this may be either a or $-a$. If we take the latter, and continue the operation as before, we shall arrive at the result $-a - b$. A similar remark holds in every other case. Take, for example, the last of those worked out in Art. 295. Here we begin by extracting the square root of x^6; this may be either x^3 or $-x^3$. If we take the latter, and continue the operation as before, we shall arrive at the result $-x^3 - 2x^2 + 2x + 1$.

297. The *fourth* root of an expression may be found by extracting the square root of the square root; similarly the *eighth* root may be found, by extracting the square root of the fourth root; and so on.

298. In Arithmetic we know that we cannot find the square root of every number *exactly;* for example, we cannot find the square root of 2 exactly. In Algebra we cannot find the square root of every proposed expression

exactly. We sometimes find such an example as the following proposed; find four terms of the square root of $1-2x$.

$$
\begin{array}{l}
1-2x\,\big(\,1-x-\frac{x^2}{2}-\frac{x^3}{2} \\
1 \\
\hline
2-x\,)\; -2x \\
\qquad\quad -2x+x^2 \\
\hline
2-2x-\frac{x^2}{2}\,)\; -x^2 \\
\qquad\qquad\quad -x^2+x^3+\frac{x^4}{4} \\
\hline
2-2x-x^2-\frac{x^3}{2}\,)\; -x^3-\frac{x^4}{4} \\
\qquad\qquad\qquad -x^3+x^4+\frac{x^5}{2}+\frac{x^6}{4} \\
\hline
\qquad\qquad\qquad\quad -\frac{5x^4}{4}-\frac{x^5}{2}-\frac{x^6}{4}
\end{array}
$$

Thus we have a remainder $-\frac{5x^4}{4}-\frac{x^5}{2}-\frac{x^6}{4}$, after finding four terms of the square root of $1-2x$; and so we know that $\left(1-x-\frac{x^2}{2}-\frac{x^3}{2}\right)^2=1-2x+\frac{5x^4}{4}+\frac{x^5}{2}+\frac{x^6}{4}$.

299. The preceding investigation of the square root of an Algebraical expression will enable us to demonstrate the rule which is given in Arithmetic for the extraction of the square root of a number.

The square root of 100 is 10, the square root of 10000 is 100, the square root of 1000000 is 1000, and so on; hence it follows that, the square root of a number less than 100 must consist of only one figure, the square root of a

number between 100 and 10000 of two places of figures, of a number between 10000 and 1000000 of three places of figures, and so on. If then a point be placed over every second figure in any number, beginning with the figure in the units' place, the number of points will shew the number of figures in the square root. Thus, for example, the square root of $4\dot{3}5\dot{6}$ consists of two figures, and the square root of $6\dot{1}1\dot{5}2\dot{4}$ consists of three figures.

300. Suppose the square root of 3249 required.

$$
\begin{array}{r|l}
 & 3\dot{2}4\dot{9}\,(50+7 \\
 & 2500 \\
\hline
100+7) & 749 \\
 & 749 \\
\hline
\end{array}
$$

Point the number according to the rule; thus it appears that the root must consist of two places of figures. Let $a+b$ denote the root, where a is the value of the figure in the tens' place, and b of that in the units' place. Then a must be the greatest multiple of ten, which has its square less than 3200; this is found to be 50. Subtract a^2, that is, the square of 50, from the given number, and the remainder is 749. Divide this remainder by $2a$, that is, by 100, and the quotient is 7, which is the value of b. Then $(2a+b)b$, that is, 107×7 or 749, is the number to be subtracted; and as there is now no remainder, we conclude that $50+7$ or 57 is the required square root.

It is stated above that a is the greatest multiple of ten which has its square less than 3200. For a evidently cannot be a *greater* multiple of ten. If possible, suppose it to be some multiple of ten *less* than this, say x; then since x is in the tens' place, and b in the units' place, $x+b$ is less than a; therefore the square of $x+b$ is less than a^2, and consequently $x+b$ is less than the true square root.

If the root consist of three places of figures, let a represent the hundreds, and b the tens; then having obtained a and b as before, let the hundreds and tens together be considered as a new value of a, and find a new value of b for the units.

301. The cyphers may be omitted for the sake of brevity, and the following rule may be obtained from the process.

```
     3249 (57
     25
     ----
107) 749
     749
     ----
```

Point every second figure, beginning with that in the units' place, and thus divide the whole number into periods. Find the greatest number whose square is contained in the first period; this is the first figure in the root; subtract its square from the first period, and to the remainder bring down the next period. Divide this quantity, omitting the last figure, by twice the part of the root already found, and annex the result to the root and also to the divisor; then multiply the divisor as it now stands by the part of the root last obtained for the subtrahend. If there be more periods to be brought down, the operation must be repeated.

302. Examples.

Extract the square root of 132496, and of 5322249.

```
     132496 (364              5322249 (2307
     9                        4
     ------                   ------
 66) 424                  43) 132
     396                      129
     ------                   ------
724) 2896               4607) 32249
     2896                     32249
```

In the first example, after the first figure of the root is found and we have brought down the remainder, we have 424; according to the rule we divide 42 by 6 to give the next figure in the root: thus apparently 7 is the next figure. But on multiplying 67 by 7 we obtain the product 469, which is greater than 424. This shews that 7 is too large for the second figure of the root, and we accordingly try 6, which succeeds. We are liable occasionally in this manner to try too large a figure, especially at the early stages of the extraction of a square root.

In the second example, the student should notice the occurrence of the cypher in the root.

303. The rule for extracting the square root of a *decimal* follows from the preceding rule. We must observe, however, that if any decimal be squared there will be an *even* number of decimal places in the result, and therefore there cannot be an exact square root of any decimal which in its simplest state has an *odd* number of decimal places.

The square root of 32·49 is one-tenth of the square root of 100 × 32·49; that is of 3249. So also the square root of ·003249, is one-thousandth of the square root of 1000000 × ·003249, that is of 3249. Thus we may deduce this rule for extracting the square root of a decimal. *Put a point over every second figure, beginning with that in the units' place and continuing both to the right and to the left of it; then proceed as in the extraction of the square root of integers, and mark off as many decimal places in the result as the number of periods in the decimal part of the proposed number.* In this rule the student should pay particular attention to the words *beginning with that in the units' place.*

304. In the extraction of the square root of an integer, if there is still a remainder after we have arrived at the figure in the units' place of the root, it indicates that the proposed number has not an exact square root. We may if we please proceed with the approximation to any desired extent, by supposing a decimal point at the end of the proposed number, and annexing any even number of cyphers, and continuing the operation. We thus obtain a decimal part to be added to the integral part already found.

Similarly, if a decimal number has no exact square root, we may annex cyphers, and proceed with the approximation to any desired extent.

305. The following is the extraction of the square root of ·4 to seven decimal places:

```
       0·4000...('6324555
        36
       ---
 123) 400
       369
      ----
1262) 3100
       2524
      -----
12644) 57600
        50576
       ------
126485) 702400
         632425
        -------
1264905) 6997500
          6324525
         --------
12649105) 67297500
           63245525
          ---------
            4051975
```

306. We now proceed to the method of extracting the cube root of a compound expression.

The cube root of $a^3+3a^2b+3ab^2+b^3$ is $a+b$; and we shall be led to a general rule for the extraction of the cube root of any compound expression by observing the manner in which $a+b$ may be derived from $a^3+3a^2b+3ab^2+b^3$.

$$
\begin{array}{l}
a^3+3a^2b+3ab^2+b^3\ (a+b \\
a^3 \\
\hline
3a^2)\ 3a^2b+3ab^2+b^3 \\
\quad\quad 3a^2b+3ab^2+b^3 \\
\hline
\end{array}
$$

Arrange the terms according to the dimensions of one letter a; then the first term is a^3, and its cube root is a, which is the first term of the required root. Subtract its cube, that is a^3, from the whole expression, and bring down the re-

mainder $3a^2b+3ab^2+b^3$. Divide $3a^2b$ by $3a^2$, and the quotient is b, which is the other term of the required root; then subtract $3a^2b+3ab^2+b^3$ from the remainder, and the whole cube of $a+b$ has been subtracted. This finishes the operation in the present case.

If there were more terms we should proceed with $a+b$ as we did formerly with a; its cube, that is $a^3+3a^2b+3ab^2+b^3$, has already been subtracted from the proposed expression, so we should divide the remainder by $3(a+b)^2$ for a new term in the root; and so on.

307. It will be convenient in extracting the cube root of more complex expressions, and of numbers, to arrange the process of the preceding Article in three columns, as follows:

$3a+b$	$3a^2$	$a^3+3a^2b+3ab^2+b^3\,(a+b$
	$(3a+b)b$	a^3
	$3a^2+3ab+b^2$	$3a^2b+3ab^2+b^3$
		$3a^2b+3ab^2+b^3$

Find the first term of the root, that is a; put a^3 under the given expression in the third column and subtract it. Put $3a$ in the first column, and $3a^2$ in the second column; divide $3a^2b$ by $3a^2$, and thus obtain the quotient b. Add b to the expression in the first column; multiply the expression now in the first column by b, and place the product in the second column, and add it to the expression already there; thus we obtain $3a^2+3ab+b^2$. Multiply this by b, and we obtain $3a^2b+3ab^2+b^3$, which is to be placed in the third column and subtracted. We have thus completed the process of subtracting $(a+b)^3$ from the original expression. If there were more terms the operation would have to be continued.

308. In continuing the operation we must add such a term to the first column, as to obtain there *three times the part of the root already found.* This is conveniently effected thus; we have already in the first column $3a+b$; place $2b$ below b and add; thus we obtain $3a+3b$, which is three times $a+b$, that is, three times the part of the root already found. Moreover, we must add such a term to the second column, as to obtain there *three times the square of the part of the root already found.* This is conveniently effected thus; we have already in the second column $(3a+b)b$, and below that $3a^2+3ab+b^2$; place b^2 below, and add the expressions in the three lines; thus we obtain $3a^2+6ab+3b^2$, which is three times $(a+b)^2$, that is three times the square of the part of the root already found.

$$\begin{array}{r} 3a+b \\ 2b \\ \hline 3a+3b \end{array}$$

$$\begin{array}{r} (3a+b)b \\ 3a^2+3ab+b^2 \\ b^2 \\ \hline 3a^2+6ab+3b^2 \end{array}$$

309. Example. Extract the cube root of

$$8x^6-36x^5+102x^4-171x^3+204x^2-144x+64.$$

$$\begin{array}{r} 6x^2-3x \\ -6x \\ \hline 6x^2-9x+4 \end{array}$$

$$\begin{array}{r} 12x^4 \\ -3x(6x^2-3x) \\ \hline 12x^4-18x^3+9x^2 \\ 9x^2 \\ \hline 12x^4-36x^3+27x^2 \\ 4(6x^2-9x+4) \\ \hline 12x^4-36x^3+51x^2-36x+16 \end{array}$$

$$\begin{array}{l} 8x^6-36x^5+102x^4-171x^3+204x^2-144x+64\ (2x^2-3x+4 \\ 8x^6 \\ \hline \quad -36x^5+102x^4-171x^3+204x^2-144x+64 \\ \quad -36x^5+\ 54x^4-\ 27x^3 \\ \hline \qquad\qquad 48x^4-144x^3+204x^2-144x+64 \\ \qquad\qquad 48x^4-144x^3+204x^2-144x+64 \end{array}$$

The cube root of $8x^6$ is $2x^2$, which will be the first term of the required root; put $8x^6$ under the given expression in the third column and subtract it. Put three times $2x^2$ in the first column, and three times the square of $2x^2$ in the second column; that is, put $6x^2$ in the first column, and $12x^4$ in the second column. Divide $-36x^5$ by $12x^4$, and thus obtain the quotient $-3x$, which will be the second term of the root; place this term in the first column, and multiply the expression now in the first column, that is $6x^2-3x$, by $-3x$; place the product under the expression in the second column, and add it to that expression; thus we obtain $12x^4-18x^3+9x^2$; multiply this by $-3x$, and place the product in the third column and subtract. Thus we have a remainder in the third column, and the part of the root already found is $2x^2-3x$. We must now adjust the first and second columns in the manner explained in Art. 308. We put twice $-3x$, that is $-6x$, in the first column, and add the two lines; thus we obtain $6x^2-9x$, which is three times the part of the root already found. We put the square of $-3x$, that is $9x^2$, in the second column, and add the last three lines in this column; thus we obtain $12x^4-36x^3+27x^2$, which is three times the square of the part of the root already found.

Now divide the remainder in the third column by the expression just obtained, and we arrive at 4 for the last term of the root, and with this we proceed as before. Place this term in the first column, and multiply the expression now in the first column, that is $6x^2-9x+4$, by 4; place the product under the expression in the second column, and add it to that expression; thus we obtain $12x^4-36x^3+51x^2-36x+16$; multiply this by 4 and place the product in the third column and subtract. As there is now no remainder we conclude that $2x^2-3x+4$ is the required cube root.

310. The preceding investigation of the cube root of an Algebraical expression will suggest a method for the extraction of the cube root of any number.

The cube root of 1000 is 10, the cube root of 1000000 is 100, and so on; hence it follows that, the cube root of

a number less than 1000 must consist of only one figure, the cube root of a number between 1000 and 1000000 of two places of figures, and so on. If then a point be placed over every third figure in any number, beginning with the figure in the units' place, the number of points will shew the number of figures in the cube root. Thus, for example, the cube root of $40\dot{5}22\dot{4}$ consists of two figures, and the cube root of $1\dot{2}81\dot{2}90\dot{4}$ consists of three figures.

Suppose the cube root of 274625 required.

180 + 5	10800	$27\dot{4}62\dot{5}$ (60 + 5
	925	216000
	11725	58625
		58625

Point the number according to the rule; thus it appears that the root must consist of two places of figures. Let $a + b$ denote the root, where a is the value of the figure in the tens' place, and b of that in the units' place. Then a must be the greatest multiple of ten which has its cube less than 274000; this is found to be 60. Place the cube of 60, that is 216000, in the third column under the given number and subtract. Place three times 60, that is 180, in the first column, and three times the square of 60, that is 10800, in the second column. Divide the remainder in the third column by the number in the second column, that is, divide 58625 by 10800; we thus obtain 5, which is the value of b. Add 5 to the first column, and multiply the sum thus formed by 5, that is, multiply 185 by 5; we thus obtain 925, which we place in the second column and add to the number already there. Thus we obtain 11725; multiply this by 5, place the product in the third column, and subtract. The remainder is zero, and therefore 65 is the required cube root.

The cyphers may be omitted for brevity, and the process will stand thus:

185	108	$27\dot{4}62\dot{5}$ (65
	925	216
	11725	58625
		58625

311. Example. Extract the cube root of 109215352.

127 }	48	109215352 (478
14 }	889 }	64
1418	5689 }	45215
	49 }	39823
	6627	5392352
	11344	5392352
	674044	

After obtaining the first two figures of the root, namely 47, we adjust the first and second columns in the manner explained in Art. 308. We place twice 7 under the first column, and add the two lines, giving 141; and we place the square of 7 under the second column, and add the last three lines, giving 6627. Then the operation is continued as before. The cube root is 478.

In the course of working this example we might have imagined that the second figure of the root would be 8 or even 9; but on trial it will be found that these numbers are too large. As in the case of the square root, we are liable occasionally to try too large a figure, especially at the early stages of the operation.

312. Example. Extract the cube root of 8653002877.

605 }	1200	8653002877 (2053
10 }	3025 }	8
6153	123025 }	653002
	25 }	615125
	126075	37877877
	18459	37877877
	12625959	

In this example the student should notice the occurrence of the cypher in the root.

313. If the root have any number of decimal places, the cube will have thrice as many; and therefore the number of decimal places in a decimal number, which is a perfect cube, and in its simplest state, will necessarily be a multiple of *three*, and the number of decimal places in the cube root will necessarily be a third of that number. Hence if the given cube number be a decimal, we place a point *over the figure in the units' place*, and over every third figure to the right and to the left of it, and proceed as in the extraction of the cube root of an integer; then the number of points in the decimal part of the proposed number will indicate the number of decimal places in the cube root.

314. Example. Extract the cube root of 14102·327296.

64	12	14102·327296(24·16
8	256	8
721	1456	6102
2	16	5824
7236	1728	278327
	721	173521
	173521	104806296
	1	104806296
	174243	
	43416	
	17467716	

315. If any number, integral or decimal, has no exact cube root, we may annex cyphers, and proceed with the approximation to the cube root to any desired extent.

The following is the extraction of the cube root of ·4 to four decimal places:

213	147	·400... (·7368
6	639	343
2196	15339	57000
12	9	46017
22088	15987	10983000
	13176	9671256
	1611876	1311744000
	36	1301484032
	1625088	10259968
	176704	
	162685504	

EXAMPLES. XXXII.

Find the value of

1. $\sqrt{(9a^4b^4)}$.
2. $\sqrt[3]{(8a^3b^3)}$.
3. $\sqrt[3]{(-64a^3b^6)}$.
4. $\sqrt[4]{(16a^4b^8c^{12})}$.
5. $\sqrt[5]{(-a^5b^{10}c^{15})}$.
6. $\sqrt{\left(\dfrac{25a^2b^2}{49c^4}\right)}$.
7. $\sqrt[3]{\left(-\dfrac{216a^3b^9}{125c^6}\right)}$.
8. $\sqrt[4]{\left(\dfrac{81a^4}{b^4c^4}\right)}$.
9. $\sqrt[5]{\left(\dfrac{a^5}{32b^{10}}\right)}$.
10. $\sqrt[6]{\left(\dfrac{64a^6b^{12}}{c^{24}}\right)}$.

Find the square roots of the following expressions:

11. $16a^2+40ab+25b^2$.
12. $49a^4-84a^2b+36b^2$.
13. $36x^6+12x^3+1$.
14. $64a^2+48abc+9b^2c^2$.
15. $\dfrac{25a^2+20ab+4b^2}{25a^2+20ac+4c^2}$.
16. $\dfrac{9x^4-24x^2+16}{4x^2-12x+9}$.

17. $x^4+2x^3+3x^2+2x+1$. 18. $1-2x+5x^2-4x^3+4x^4$.

19. $x^4+6x^3+25x^2+48x+64$. 20. x^4-4x^3+8x+4.

21. $1-4x+10x^2-12x^3+9x^4$.

22. $4x^8-4x^6-7x^4+4x^2+4$.

23. $x^4-2ax^3+5a^2x^2-4a^3x+4a^4$.

24. $x^4-2ax^3+(a^2+2b^2)x^2-2ab^2x+b^4$.

25. $x^6-12x^5+60x^4-160x^3+240x^2-192x+64$.

26. $x^6+4ax^5-10a^3x^3+4a^5x+a^6$.

27. $1-2x+3x^2-4x^3+5x^4-4x^5+3x^6-2x^7+x^8$.

28. $\frac{4x^2}{9y^2}-\frac{x}{z}-\frac{16x^2}{15yz}+\frac{9y^2}{16z^2}+\frac{6xy}{5z^2}+\frac{16x^2}{25z^2}$.

Find the fourth roots of the following expressions:

29. $1+4x+6x^2+4x^3+x^4$.

30. $16x^4-96x^3y+216x^2y^2-216xy^3+81y^4$.

31. $1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8$.

32. $\{x^4-2(a+b)x^3+(a^2+4ab+b^2)x^2-2ab(a+b)x+a^2b^2\}^2$.

Find the eighth roots of the following expressions:

33. $x^8+8x^7+28x^6+56x^5+70x^4+56x^3+28x^2+8x+1$.

34. $\{x^4-2x^3y+3x^2y^2-2xy^3+y^4\}^4$.

Find the square roots of the following numbers:

35. 1156. 36. 2025. 37. 3721. 38. 5184.

39. 7569. 40. 9801. 41. 15129. 42. 103041.

43. 165649. 44. 3080·25. 45. 41·2164.

46. ·835396. 47. 1522756. 48. 29376400.

49. 384524·01. 50. 4981·5364. 51. 64·128064.

52. ·24373969. 53. 144168049. 54. 254076·4836.

55. 3·25513764. 56. 4·54499761.

57. ·5687573056. 58. 196540602241.

Extract the square root of each of the following num-
rs to five places of decimals:

59. ·9. 60. 6·21. 61. ·43. 62. ·00852.

63. 17. 64. 129. 65. 347·259. 66. 14295·387.

Find the cube roots of the following expressions:

67. $8x^3 + 36x^2y + 54xy^2 + 27y^3$.

68. $1728x^6 + 1728x^4y^3 + 576x^2y^6 + 64y^9$.

69. $x^3 - 3x^2(a+b) + 3x(a+b)^2 - (a+b)^3$.

70. $x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1$.

71. $x^6 - 3ax^5 + 5a^3x^3 - 3a^5x - a^6$.

72. $8x^6 + 48cx^5 + 60c^2x^4 - 80c^3x^3 - 90c^4x^2 + 108c^5x - 27c^6$.

73. $1 - 9x + 39x^2 - 99x^3 + 156x^4 - 144x^5 + 64x^6$.

74. $1 - 3x + 6x^2 - 10x^3 + 12x^4 - 12x^5 + 10x^6 - 6x^7 + 3x^8 - x^9$.

Find the sixth roots of the following expressions:

75. $1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$.

76. $729x^6 - 1458x^5 + 1215x^4 - 540x^3 + 135x^2 - 18x + 1$.

Find the cube roots of the following numbers:

77. 19683. 78. 42875. 79. 157464.

80. 226981. 81. 681472. 82. 778688.

83. 2628072. 84. 3241792. 85. 54010152.

86. 60236·288. 87. 191·102976. 88. ·220348864

89. 1371330631. 90. 20910518875.

91. 91398648463125. 92. 5340104393239.

XXXIII. Indices.

316. We have defined an *index* or *exponent* in Art. 16, and, according to that definition, an index has hitherto always been a positive whole number. We are now about to extend the definition of an index, by explaining the meaning of fractional indices and of negative indices.

317. *If* m *and* n *are any positive whole numbers* $a^m \times a^n = a^{m+n}$.

The truth of this statement has already been shewn in Art. 59, but it is convenient to repeat the demonstration here.

$$a^m = a \times a \times a \times \ldots\ldots \text{to } m \text{ factors, by Art. 16,}$$

$$a^n = a \times a \times a \times \ldots\ldots \text{to } n \text{ factors, by Art. 16;}$$

therefore

$$a^m \times a^n = a \times a \times a \times \ldots \times a \times a \times a \times \ldots \text{ to } m+n \text{ factors}$$
$$= a^{m+n}, \text{ by Art. 16.}$$

In like manner, if p is also a positive whole number,

$$a^m \times a^n \times a^p = a^{m+n} \times a^p = a^{m+n+p};$$

and so on.

318. If m and n are positive whole numbers, and m greater than n, we have by Art. 317

$$a^{m-n} \times a^n = a^{m-n+n} = a^m;$$

therefore
$$\frac{a^m}{a^n} = a^{m-n}.$$

This also has been already shewn; see Art. 72.

319. As fractional indices and negative indices have not yet been defined, we are at liberty to give what definitions we please to them; and it is found convenient to

give such definitions to them as will make the important relation $a^m \times a^n = a^{m+n}$ *always true, whatever* m *and* n *may be.*

For example; required the meaning of $a^{\frac{1}{2}}$.

By supposition we are to have $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^1 = a$. Thus $a^{\frac{1}{2}}$ must be such a number that if it be multiplied by itself the result is a; and the *square root* of a is by definition such a number; therefore $a^{\frac{1}{2}}$ must be equivalent to the square root of a, that is, $a^{\frac{1}{2}} = \sqrt{a}$.

Again; required the meaning of $a^{\frac{1}{3}}$.

By supposition we are to have

$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a.$$

Hence, as before, $a^{\frac{1}{3}}$ must be equivalent to the cube root of a, that is $a^{\frac{1}{3}} = \sqrt[3]{a}$.

Again; required the meaning of $a^{\frac{3}{4}}$.

By supposition, $a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} = a^3$;

therefore $$a^{\frac{3}{4}} = \sqrt[4]{a^3}.$$

These examples would enable the student to understand what is meant by any fractional exponent; but we will give the definition in general symbols in the next two Articles.

320. *Required the meaning of* $a^{\frac{1}{n}}$ *where* n *is any positive whole number.*

By supposition,

$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots \text{to } n \text{ factors} = a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\ldots \text{to } n \text{ terms}} = a^1 = a;$$

therefore $a^{\frac{1}{n}}$ must be equivalent to the n^{th} root of a,

that is, $$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

321. *Required the meaning of* $a^{\frac{m}{n}}$ *where* m *and* n *are any positive whole numbers.*

By supposition,

$$a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times \ldots \text{ to } n \text{ factors} = a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}+\ldots \text{ to } n \text{ terms}} = a^m;$$

therefore $a^{\frac{m}{n}}$ must be equivalent to the n^{th} root of a^m,

that is, $$a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

Hence $a^{\frac{m}{n}}$ means the n^{th} root of the m^{th} power of a; that is, in a fractional index the numerator denotes a power and the denominator a root.

322. We have thus assigned a meaning to any positive index, whether whole or fractional; it remains to assign a meaning to negative indices.

For example, required the meaning of a^{-2}.

By supposition, $$a^3 \times a^{-2} = a^{3-2} = a^1 = a,$$

therefore $$a^{-2} = \frac{a}{a^3} = \frac{1}{a^2}.$$

We will now give the definition in general symbols.

323. *Required the meaning of* a^{-n}; *where* n *is any positive number whole or fractional.*

By supposition, whatever m may be, we are to have

$$a^m \times a^{-n} = a^{m-n}.$$

Now we may suppose m positive and greater than n, and then, by what has gone before, we have

$$a^{m-n} \times a^n = a^m; \qquad \text{and therefore } a^{m-n} = \frac{a^m}{a^n}.$$

Therefore $$a^m \times a^{-n} = \frac{a^m}{a^n};$$

therefore $$a^{-n} = \frac{1}{a^n}.$$

In order to express this in words we will define the word *reciprocal.* One quantity is said to be the *reciprocal* of another when the product of the two is equal to unity; thus, for example, x is the *reciprocal* of $\frac{1}{x}$.

Hence a^{-n} is the reciprocal of a^n; or we may put this result symbolically in any of the following ways,

$$a^{-n} = \frac{1}{a^n}, \qquad a^n = \frac{1}{a^{-n}}, \quad a^n \times a^{-n} = 1.$$

324. It will follow from the meaning which has been given to a negative index that $a^m \div a^n = a^{m-n}$ when m is less than n, as well as when m is greater than n. For suppose m less than n; we have

$$a^m \div a^n = \frac{a^m}{a^n} = \frac{1}{a^{n-m}} = a^{-(n-m)} = a^{m-n}.$$

Suppose $m = n$; then $a^m \div a^n$ is obviously $= 1$; and $a^{m-n} = a^0$. The last symbol has not hitherto received a meaning, so that we are at liberty to give it the meaning which naturally presents itself; hence we may say that $a^0 = 1$.

325. In order to form a complete theory of Indices it would be necessary to give demonstrations of several propositions which will be found in the larger Algebra. But these propositions follow so naturally from the definitions and the properties of fractions, that the student will not find any difficulty in the simple cases which will come before him. We shall therefore refer for the complete theory to the larger Algebra, and only give here some examples as specimens.

326. If m and n are positive whole numbers we know that $(a^m)^n = a^{mn}$; see Art. 279. Now this result will also hold when m and n are not positive whole numbers. For example,

$$(a^{\frac{1}{3}})^{\frac{1}{4}} = a^{\frac{1}{12}}.$$

For let $(a^{\frac{1}{3}})^{\frac{1}{4}} = x$; then by raising both sides to the fourth power we have $a^{\frac{1}{3}} = x^4$; then by raising both sides

to the third power we have $a = x^{12}$; therefore $x = a^{\frac{1}{12}}$, which was to be shewn.

327. If n is a positive whole number we know that $a^n \times b^n = (ab)^n$. This result will also hold when n is not a positive whole number. For example, $a^{\frac{1}{3}} \times b^{\frac{1}{3}} = (ab)^{\frac{1}{3}}$. For if we raise each side to the third power, we obtain in each case ab; so that each side is the cube root of ab.

In like manner we have

$$a^{\frac{1}{n}} \times b^{\frac{1}{n}} \times c^{\frac{1}{n}} \times \ldots = (abc\ldots)^{\frac{1}{n}}.$$

Suppose now that there are m of these quantities $a, b, c, \ldots$, and that all the rest are equal to a; thus we obtain

$$\left(a^{\frac{1}{n}}\right)^m = \left(a^m\right)^{\frac{1}{n}}; \text{ that is, } (\sqrt[n]{a})^m = \sqrt[n]{a^m}.$$

Thus the m^{th} power of the n^{th} root of a is equal to the n^{th} root of the m^{th} power of a.

328. Since a fraction may take different forms without any change in its value, we may expect to be able to give different forms to a quantity with a fractional index, without altering the value of the quantity. Thus, for example, since $\frac{2}{3} = \frac{4}{6}$ we may expect that $a^{\frac{2}{3}} = a^{\frac{4}{6}}$; and this is the case. For if we raise each side to the sixth power, we obtain a^4; that is, each side is the sixth root of a^4.

329. We will now give some examples of Algebraical operations involving fractional and negative exponents.

Multiply $a^{\frac{2}{3}} b^{\frac{3}{4}} c^{\frac{1}{3}}$ by $a^{\frac{1}{2}} b^{\frac{1}{3}} c^{\frac{2}{3}}$.

$$\frac{2}{3} + \frac{1}{2} = \frac{7}{6}, \qquad \frac{3}{4} + \frac{1}{3} = \frac{13}{12}, \qquad \frac{1}{3} + \frac{2}{3} = 1,$$

therefore $$a^{\frac{2}{3}} b^{\frac{3}{4}} c^{\frac{1}{3}} \times a^{\frac{1}{2}} b^{\frac{1}{3}} c^{\frac{2}{3}} = a^{\frac{7}{6}} b^{\frac{13}{12}} c.$$

Divide $x^{\frac{3}{4}}y^{\frac{2}{3}}$ by $x^{\frac{1}{2}}y^{\frac{1}{6}}$.

$$\frac{3}{4}-\frac{1}{2}=\frac{1}{4}, \qquad \frac{2}{3}-\frac{1}{6}=\frac{1}{2};$$

therefore $$x^{\frac{3}{4}}y^{\frac{2}{3}} \div x^{\frac{1}{2}}y^{\frac{1}{6}} = x^{\frac{1}{4}}y^{\frac{1}{2}}.$$

Multiply $\quad x+x^{\frac{1}{3}}+x^{-\frac{1}{3}}$ by $x^{\frac{1}{3}}+x^{-\frac{1}{3}}-x^{-1}$.

$$\begin{array}{llllll}
x & +x^{\frac{1}{3}} & +x^{-\frac{1}{3}} & & & \\
x^{\frac{1}{3}} & +x^{-\frac{1}{3}} & -x^{-1} & & & \\
\hline
x^{\frac{4}{3}} & +x^{\frac{2}{3}} & +1 & & & \\
 & x^{\frac{2}{3}} & +1 & +x^{-\frac{2}{3}} & & \\
 & & -1 & -x^{-\frac{2}{3}} & -x^{-\frac{4}{3}} & \\
\hline
x^{\frac{4}{3}} & +2x^{\frac{2}{3}} & +1 & & -x^{-\frac{4}{3}} & \\
\hline
\end{array}$$

Here in the first line $x^{\frac{1}{3}} \times x = x^{\frac{1}{3}+1} = x^{\frac{4}{3}}$, $x^{\frac{1}{3}} \times x^{\frac{1}{3}} = x^{\frac{2}{3}}$, $x^{\frac{1}{3}} \times x^{-\frac{1}{3}} = x^0 = 1$; and so on.

Divide

$$x^{\frac{1}{2}}-3x^{\frac{1}{3}}y^{-\frac{1}{6}}+3x^{\frac{1}{6}}y^{-\frac{1}{3}}-y^{-\frac{1}{2}} \text{ by } x^{\frac{1}{3}}-2x^{\frac{1}{6}}y^{-\frac{1}{6}}+y^{-\frac{1}{3}}.$$

$$\begin{array}{l}
x^{\frac{1}{3}}-2x^{\frac{1}{6}}y^{-\frac{1}{6}}+y^{-\frac{1}{3}}\,\big)\,x^{\frac{1}{2}}-3x^{\frac{1}{3}}y^{-\frac{1}{6}}+3x^{\frac{1}{6}}y^{-\frac{1}{3}}-y^{-\frac{1}{2}}\,\big(x^{\frac{1}{6}}-y^{-\frac{1}{6}} \\
\qquad\qquad\qquad\qquad\quad x^{\frac{1}{2}}-2x^{\frac{1}{3}}y^{-\frac{1}{6}}+\ x^{\frac{1}{6}}y^{-\frac{1}{3}} \\
\hline
\qquad\qquad\qquad\qquad\qquad\ -\ x^{\frac{1}{3}}y^{-\frac{1}{6}}+2x^{\frac{1}{6}}y^{-\frac{1}{3}}-y^{-\frac{1}{2}} \\
\qquad\qquad\qquad\qquad\qquad\ -\ x^{\frac{1}{3}}y^{-\frac{1}{6}}+2x^{\frac{1}{6}}y^{-\frac{1}{3}}-y^{-\frac{1}{2}} \\
\hline
\end{array}$$

EXAMPLES. XXXIII.

Find the value of

1. $9^{-\frac{1}{2}}$. 2. $4^{-\frac{3}{2}}$. 3. $(100)^{-\frac{1}{2}}$. 4. $(1000)^{\frac{2}{3}}$. 5. $(81)^{-\frac{3}{4}}$.

Simplify

6. $(a^2)^{-3}$. 7. $(a^{-2})^{-3}$. 8. $\sqrt{a^{-4}}$. 9. $\sqrt[3]{a^{-3}}$.

10. $a^{\frac{1}{2}} \times a^{\frac{1}{3}} \times a^{-\frac{1}{4}}$.

Multiply

11. $x^{\frac{3}{4}} + y^{\frac{3}{4}}$ by $x^{\frac{3}{4}} - y^{\frac{3}{4}}$. 12. $a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.

13. $x + x^{\frac{1}{2}} + 2$ by $x + x^{\frac{1}{2}} - 2$.

14. $x^4 + x^2 + 1$ by $x^{-4} - x^{-2} + 1$.

15. $a^{-\frac{2}{3}} + a^{-\frac{1}{3}} + 1$ by $a^{-\frac{1}{3}} - 1$.

16. $a^{\frac{4}{3}} - 2 + a^{-\frac{4}{3}}$ by $a^{\frac{2}{3}} - a^{-\frac{2}{3}}$.

17. $a + a^{\frac{1}{2}}b^{\frac{1}{2}} - x^{\frac{1}{3}}y^{\frac{2}{3}}$ by $a + a^{\frac{1}{2}}b^{\frac{1}{2}} + x^{\frac{1}{3}}y^{\frac{2}{3}}$.

18. $x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}$ by $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$.

Divide

19. $x^{\frac{2}{3}} - y^{\frac{2}{3}}$ by $x^{\frac{1}{6}} - y^{\frac{1}{6}}$. 20. $a - b$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.

21. $64x^{-1} + 27y^{-2}$ by $4x^{-\frac{1}{3}} + 3y^{-\frac{2}{3}}$.

22. $x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}$ by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$.

23. $a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$ by $a^{\frac{1}{3}} + a^{\frac{1}{6}}b^{\frac{1}{6}} + b^{\frac{1}{3}}$.

24. $a^{\frac{2}{3}} + b^{\frac{2}{3}} - c^{\frac{2}{3}} + 2a^{\frac{1}{3}}b^{\frac{1}{3}}$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}}$.

25. $x^{\frac{3}{4}} - 2a^{\frac{3}{2}}x^{\frac{3}{8}} + a^3$ by $x^{\frac{1}{4}} - 2a^{\frac{1}{2}}x^{\frac{1}{8}} + a$.

26. $x^{\frac{1}{2}} - 4x^{\frac{3}{8}}y^{\frac{1}{8}} + 6x^{\frac{1}{4}}y^{\frac{1}{4}} - 4x^{\frac{1}{8}}y^{\frac{3}{8}} + y^{\frac{1}{2}}$ by $x^{\frac{1}{4}} - 2x^{\frac{1}{8}}y^{\frac{1}{8}} + y^{\frac{1}{4}}$.

Find the square roots of the following expressions:

27. $x^{\frac{1}{2}} - 4 + 4x^{-\frac{1}{2}}$. 28. $(x + x^{-1})^2 - 4(x - x^{-1})$.

29. $x^{\frac{5}{3}} - 4x^{\frac{4}{3}} + 2x^{\frac{7}{6}} + 4x - 4x^{\frac{5}{6}} + x^{\frac{2}{3}}$.

30. $4x^{\frac{3}{2}} - 12x^{\frac{3}{4}} + 25 - 24x^{-\frac{3}{4}} + 16x^{-\frac{3}{2}}$.

XXXIV. *Surds.*

330. When a root of a number cannot be exactly obtained it is called an *irrational quantity*, or a *surd*. Thus, for example, the following are surds;

$$\sqrt{5},\quad \sqrt{\frac{2}{3}},\quad \sqrt[3]{4},\quad \sqrt[3]{\frac{3}{4}},\quad \sqrt[4]{7}.$$

And if a root of an algebraical expression cannot be denoted without the use of a fractional index, it is also called an *irrational* quantity or a *surd*. Thus, for example, the following are surds;

$$\sqrt{a},\quad \sqrt{\frac{a}{b}},\quad \sqrt{(a^2+ab+b^2)},\quad \sqrt[3]{a^2},\quad \sqrt[3]{(a^3+b^3)}.$$

The rules for operations with surds follow from the propositions of the preceding Chapter; and the present Chapter consists almost entirely of the application of those propositions to arithmetical examples.

331. Numbers or expressions may occur in the *form* of surds, which are not *really* surds. Thus, for example, $\sqrt{9}$ is in the form of a surd, but it is not really a surd, for $\sqrt{9}=3$; and $\sqrt{(a^2+2ab+b^2)}$ is in the form of a surd, but it is not really a surd, for $\sqrt{(a^2+2ab+b^2)}=a+b$.

332. It is often convenient to put a rational quantity into the form of an assigned surd; to do this we raise the quantity to the power corresponding to the root indicated by the surd, and prefix the radical sign. For example,

$$3=\sqrt{3^2}=\sqrt{9};\quad 4=\sqrt[3]{4^3}=\sqrt[3]{64};\quad a=\sqrt[4]{a^4};\quad a+b=\sqrt[5]{(a+b)^5}.$$

333. The product of a rational quantity and a surd may be expressed as an entire surd, by reducing the rational quantity to the form of the surd, and then multiplying; see Art. 327. For example, $3\sqrt{2}=\sqrt{9}\times\sqrt{2}=\sqrt{18}$;

$$2\sqrt[3]{4}=\sqrt[3]{8}\times\sqrt[3]{4}=\sqrt[3]{32};\quad a\sqrt{b}=\sqrt{a^2}\times\sqrt{b}=\sqrt{(a^2b)}.$$

334. Conversely, an entire surd may be expressed as the product of a rational quantity and a surd, if the root of one factor can be extracted.

For example, $\sqrt{32} = \sqrt{(16 \times 2)} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$;

$$\sqrt[3]{48} = \sqrt[3]{(8 \times 6)} = \sqrt[3]{8} \times \sqrt[3]{6} = 2\sqrt[3]{6};$$

$$\sqrt[3]{(a^3b^2)} = \sqrt[3]{a^3} \times \sqrt[3]{b^2} = a\sqrt[3]{b^2}.$$

335. A surd fraction can be transformed into an equivalent expression with the surd part integral.

For example, $\sqrt{\frac{3}{8}} = \sqrt{\frac{3 \times 2}{8 \times 2}} = \sqrt{\frac{6}{16}} = \frac{\sqrt{6}}{4}$;

$$\sqrt[3]{\frac{2}{3}} = \sqrt[3]{\frac{2 \times 9}{3 \times 9}} = \sqrt[3]{\frac{18}{27}} = \frac{\sqrt[3]{18}}{3}.$$

336. Surds which have not the same index can be transformed into equivalent surds which have; see Art. 327.

For example, take $\sqrt{5}$ and $\sqrt[3]{11}$: $\sqrt{5} = 5^{\frac{1}{2}}$, $\sqrt[3]{11} = (11)^{\frac{1}{3}}$;

$$5^{\frac{1}{2}} = 5^{\frac{3}{6}} = \sqrt[6]{5^3} = \sqrt[6]{125}, \quad (11)^{\frac{1}{3}} = 11^{\frac{2}{6}} = \sqrt[6]{(11)^2} = \sqrt[6]{121}.$$

337. We may notice an application of the preceding Article. Suppose we wish to know which is the greater, $\sqrt{5}$ or $\sqrt[3]{11}$. When we have reduced them to the same index we see that the former is the greater, because 125 is greater than 121.

338. Surds are said to be *similar* when they have, or can be reduced to have, the same irrational factors.

Thus $4\sqrt{7}$ and $5\sqrt{7}$ are similar surds; $5\sqrt[3]{2}$ and $4\sqrt[3]{16}$ are also similar surds, for $4\sqrt[3]{16} = 8\sqrt[3]{2}$.

339. To add or subtract similar surds, add or subtract their coefficients, and affix to the result the common irrational factor.

For example, $\sqrt{12} + \sqrt{75} - \sqrt{48} = 2\sqrt{3} + 5\sqrt{3} - 4\sqrt{3}$

$$= (2 + 5 - 4)\sqrt{3} = 3\sqrt{3}.$$

$$\frac{2}{3}\sqrt[3]{\frac{3}{2}} + \frac{1}{4}\sqrt[3]{\frac{256}{9}} = \frac{2}{3}\sqrt[3]{\frac{12}{8}} + \frac{1}{4}\sqrt[3]{\frac{64 \times 12}{27}}$$

$$= \frac{2}{3}\frac{\sqrt[3]{12}}{2} + \frac{1}{4}\frac{4\sqrt[3]{12}}{3} = \frac{2\sqrt[3]{12}}{3}.$$

340. To multiply simple surds which have the same index, multiply separately the rational factors and the irrational factors.

For example, $3\sqrt{2} \times \sqrt{3} = 3\sqrt{6}$; $4\sqrt{5} \times 7\sqrt{6} = 28\sqrt{30}$;

$$2\sqrt[3]{4} \times 3\sqrt[3]{2} = 6\sqrt[3]{8} = 6 \times 2 = 12.$$

341. To multiply simple surds which have not the same index, reduce them to equivalent surds which have the same index, and then proceed as before.

For example, multiply $4\sqrt{5}$ by $2\sqrt[3]{11}$.

By Art. 336 $\sqrt{5} = \sqrt[6]{125}$, $\sqrt[3]{11} = \sqrt[6]{121}$.

Hence the product is $8\sqrt[6]{(125 \times 121)}$, that is, $8\sqrt[6]{15125}$.

342. The multiplication of compound surds is performed like the multiplication of compound algebraical expressions.

For example, $(6\sqrt{3} - 5\sqrt{2}) \times (2\sqrt{3} + 3\sqrt{2})$

$$= 36 + 18\sqrt{6} - 10\sqrt{6} - 30 = 6 + 8\sqrt{6}.$$

343. Division by a simple surd is performed by a rule like that for multiplication by a simple surd; the result may be simplified by Art. 335.

For example, $3\sqrt{2} \div 4\sqrt{3} = \dfrac{3\sqrt{2}}{4\sqrt{3}} = \dfrac{3}{4}\sqrt{\dfrac{2}{3}} = \dfrac{3}{4}\sqrt{\dfrac{6}{9}} = \dfrac{\sqrt{6}}{4}$;

$$4\sqrt{5} \div 2\sqrt[3]{11} = \frac{4\sqrt{5}}{2\sqrt[3]{11}} = \frac{2\sqrt[6]{125}}{\sqrt[6]{121}} = 2\sqrt[6]{\frac{125}{121}} = 2\sqrt[6]{\frac{125 \times (11)^4}{121 \times (11)^4}}$$

$$= \frac{2\sqrt[6]{1830125}}{11}.$$

The student will observe that by the aid of Art. 335 the results are put in forms which are more convenient for numerical application; thus, if we have to find the approximate numerical value of $3\sqrt{2} \div 4\sqrt{3}$, the easiest method is to extract the square root of 6, and divide the result by 4.

344. The only case of division by a compound surd which is of any importance is that in which the divisor is the sum or difference of two *quadratic* surds, that is, surds involving square roots. The division is practically effected by an important process which is called *rationalising the denominator of a fraction.* For example, take the fraction $\frac{4}{5\sqrt{2}+2\sqrt{3}}$; if we multiply both numerator and denominator of this fraction by $5\sqrt{2}-2\sqrt{3}$, the value of the fraction is not altered, while its *denominator is made rational;*

thus $$\frac{4}{5\sqrt{2}+2\sqrt{3}}=\frac{4(5\sqrt{2}-2\sqrt{3})}{(5\sqrt{2}+2\sqrt{3})(5\sqrt{2}-2\sqrt{3})}$$

$$=\frac{4(5\sqrt{2}-2\sqrt{3})}{50-12}=\frac{10\sqrt{2}-4\sqrt{3}}{19}.$$

Similarly, $$\frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}-\sqrt{2}}=\frac{(\sqrt{3}+\sqrt{2})(2\sqrt{3}+\sqrt{2})}{(2\sqrt{3}-\sqrt{2})(2\sqrt{3}+\sqrt{2})}$$

$$=\frac{8+3\sqrt{6}}{12-2}=\frac{8+3\sqrt{6}}{10}.$$

345. We shall now shew how to find the square root of a binomial expression, one of whose terms is a quadratic surd. Suppose, for example, that we require the square root of $7+4\sqrt{3}$. Since $(\sqrt{x}+\sqrt{y})^2=x+y+2\sqrt{(xy)}$, it is obvious that if we find values of x and y from $x+y=7$, and $2\sqrt{(xy)}=4\sqrt{3}$, then the square root of $7+4\sqrt{3}$ will be $\sqrt{x}+\sqrt{y}$. We may arrange the whole process thus:

Suppose $\quad \sqrt{(7+4\sqrt{3})}=\sqrt{x}+\sqrt{y};$

square, $\quad 7+4\sqrt{3}=x+y+2\sqrt{(xy)}.$

Assume $x+y=7$, then $2\sqrt{(xy)}=4\sqrt{3}$;

square, and subtract, $(x+y)^2-4xy=49-48=1$,

that is, $(x-y)^2=1$, therefore $x-y=1$.

Since $x+y=7$ and $x-y=1$, we have $x=4$, $y=3$;

therefore $\quad \sqrt{(7+4\sqrt{3})}=\sqrt{4}+\sqrt{3}=2+\sqrt{3}.$

Similarly, $\quad \sqrt{(7-4\sqrt{3})}=2-\sqrt{3}.$

Examples. XXXIV.

Simplify

1. $3\sqrt{2}+4\sqrt{8}-\sqrt{32}$.

2. $2\sqrt[3]{4}+5\sqrt[3]{32}-\sqrt[3]{108}$.

3. $2\sqrt{3}+3\sqrt{(1\frac{1}{3})}-\sqrt{(5\frac{1}{3})}$.

4. $\frac{1}{\sqrt[3]{2}}-\frac{1}{\sqrt[3]{16}}$.

Multiply

5. $\sqrt{5}+\sqrt{(1\frac{1}{4})}-\frac{1}{\sqrt{5}}$ by $\sqrt{3}$.

6. $\sqrt[3]{4}-\frac{1}{\sqrt[3]{16}}+\frac{1}{\sqrt[3]{2}}$ by $\sqrt[3]{4}$.

7. $1+\sqrt{3}-\sqrt{2}$ by $\sqrt{6}-\sqrt{2}$.

8. $\sqrt{3}+\sqrt{2}$ by $\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{2}}$.

Rationalise the denominators of the following fractions:

9. $\frac{3+\sqrt{2}}{2-\sqrt{2}}$.

10. $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

11. $\frac{2\sqrt{5}+\sqrt{3}}{3\sqrt{5}+2\sqrt{3}}$.

12. $\frac{2\sqrt{3}+3\sqrt{2}}{3\sqrt{3}-2\sqrt{5}}$.

Extract the square root of

13. $14+6\sqrt{5}$.

14. $16-6\sqrt{7}$.

15. $8+4\sqrt{3}$.

16. $4-\sqrt{15}$.

Simplify

17. $\frac{1}{\sqrt{(5+\sqrt{24})}}$.

18. $\frac{1}{\sqrt{(7-4\sqrt{3})}}$.

19. $\frac{\sqrt{(12+6\sqrt{3})}}{1+\sqrt{3}}$.

20. $\sqrt{(3+\sqrt{5})}+\sqrt{(3-\sqrt{5})}$.

XXXV. *Ratio.*

346. Ratio is the relation which one quantity bears to another with respect to magnitude, the comparison being made by considering what multiple, part, or parts, the first is of the second.

Thus, for example, in comparing 6 with 3, we observe that 6 has a certain magnitude with respect to 3, which it contains twice; again, in comparing 6 with 2, we see that 6 has now a different *relative* magnitude, for it contains 2 three times; or 6 is greater when compared with 2 than it is when compared with 3.

347. The ratio of a to b is usually expressed by two points placed between them, thus, $a : b$; and the former is called the *antecedent* of the ratio, and the latter the *consequent* of the ratio.

348. A ratio is measured by the fraction which has for its numerator the antecedent of the ratio, and for its denominator the consequent of the ratio. Thus the ratio of a to b is measured by $\frac{a}{b}$; then for shortness we may say that the ratio of a to b is equal to $\frac{a}{b}$ or is $\frac{a}{b}$.

349. Hence we may say that the ratio of a to b is equal to the ratio of c to d, when $\frac{a}{b} = \frac{c}{d}$.

350. *If the terms of a ratio be multiplied or divided by the same quantity the ratio is not altered.*

For $\frac{a}{b} = \frac{ma}{mb}$ (Art. 135).

351. We *compare* two or more ratios by reducing the fractions which measure these ratios to a common denominator. Thus, suppose one ratio to be that of a to b,

and another ratio to be that of c to d; then the first ratio $\frac{a}{b}=\frac{ad}{bd}$, and the second ratio $\frac{c}{d}=\frac{bc}{bd}$.

Hence the first ratio is greater than, equal to, or less than the second ratio, according as ad is greater than, equal to, or less than bc.

352. A ratio is called a ratio of *greater inequality*, of *less inequality*, or of *equality*, according as the antecedent is *greater* than, *less* than, or *equal* to the consequent.

353. *A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding any number to both terms of the ratio.*

Let the ratio be $\frac{a}{b}$, and let a new ratio be formed by adding x to both terms of the original ratio; then $\frac{a+x}{b+x}$ is greater or less than $\frac{a}{b}$, according as $b(a+x)$ is greater or less than $a(b+x)$; that is, according as bx is greater or less than ax, that is, according as b is greater or less than a.

354. *A ratio of greater inequality is increased, and a ratio of less inequality is diminished, by taking from both terms of the ratio any number which is less than each of those terms.*

Let the ratio be $\frac{a}{b}$, and let a new ratio be formed by taking x from both terms of the original ratio; then $\frac{a-x}{b-x}$ is greater or less than $\frac{a}{b}$, according as $b(a-x)$ is greater or less than $a(b-x)$; that is, according as bx is less or greater than ax, that is, according as b is less or greater than a.

355. If the antecedents of any ratios be multiplied together, and also the consequents, a new ratio is obtained which is said to be *compounded* of the former ratios. Thus

the ratio $ac : bd$ is said to be compounded of the two ratios $a : b$ and $c : d$.

When the ratio $a : b$ is compounded with itself the resulting ratio is $a^2 : b^2$; this ratio is sometimes called the *duplicate* ratio of $a : b$. And the ratio $a^3 : b^3$ is sometimes called the *triplicate* ratio of $a : b$.

356. The following is a very important theorem concerning equal ratios.

Suppose that $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then each of these ratios

$$= \left(\frac{pa^n + qc^n + re^n}{pb^n + qd^n + rf^n}\right)^{\frac{1}{n}}$$

where p, q, r, n are any numbers whatever.

For let $k = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}$; then

$$kb = a, \quad kd = c, \quad kf = e;$$

therefore $$p(kb)^n + q(kd)^n + r(kf)^n = pa^n + qc^n + re^n;$$

therefore $$k^n = \frac{pa^n + qc^n + re^n}{pb^n + qd^n + rf^n};$$

therefore $$k = \left(\frac{pa^n + qc^n + re^n}{pb^n + qd^n + rf^n}\right)^{\frac{1}{n}}.$$

The same mode of demonstration may be applied, and a similar result obtained when there are more than *three* ratios given equal.

As a particular example we may suppose $n = 1$, then we see that if $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, each of these ratios is equal to $\frac{pa + qc + re}{pb + qd + rf}$; and then as a special case we may suppose $p = q = r$, so that each of the given equal ratios is equal to $\frac{a + c + e}{b + d + f}$.

EXAMPLES. XXXV.

1. Find the ratio of fourteen shillings to three guineas.

2. Arrange the following ratios in the order of magnitude; 3 : 4, 7 : 12, 8 : 9, 2 : 3, 5 : 8.

3. Find the ratio compounded of 4 : 15 and 25 : 36.

4. Two numbers are in the ratio of 2 to 3, and if 7 be added to each the ratio is that of 3 to 4: find the numbers.

5. Two numbers are in the ratio of 4 to 5, and if 6 be taken from each the ratio is that of 3 to 4: find the numbers.

6. Two numbers are in the ratio of 5 to 8; if 8 be added to the less number, and 5 taken from the greater number, the ratio is that of 28 to 27: find the numbers.

7. Find the number which added to each term of the ratio 5 : 3 makes it three-fourths of what it would have become if the same number had been taken from each term.

8. Find two numbers in the ratio of 2 to 3, such that their difference has to the difference of their squares the ratio of 1 to 25.

9. Find two numbers in the ratio of 3 to 4, such that their sum has to the sum of their squares the ratio of 7 to 50.

10. Find two numbers in the ratio of 5 to 6, such that their sum has to the difference of their squares the ratio of 1 to 7.

11. Find x so that the ratio $x : 1$ may be the duplicate of the ratio $8 : x$.

12. Find x so that the ratio $a-x : b-x$ may be the duplicate of the ratio $a : b$.

13. A person has 200 coins consisting of guineas, half-sovereigns, and half-crowns; the sums of money in guineas, half-sovereigns, and half-crowns are as 14 : 8 : 3; find the numbers of the different coins.

14. If $b-a : b+a = 4a-b : 6a-b$, find $a : b$.

15. If $\dfrac{l}{a-b} = \dfrac{m}{b-c} = \dfrac{n}{c-a}$, then $l+m+n=0$.

XXXVI. *Proportion.*

357. Four numbers are said to be proportional when the first is the same multiple, part, or parts of the second as the third is of the fourth; that is when $\frac{a}{b} = \frac{c}{d}$ the four numbers a, b, c, d are called proportionals. This is usually expressed by saying that a is to b as c is to d; and it is represented thus $a : b :: c : d$, or thus $a : b = c : d$.

The terms a and d are called the *extremes*, and b and c the *means*.

358. Thus when two ratios are equal, the four numbers which form the ratios are called proportionals; and the present Chapter is devoted to the subject of two equal ratios.

359. *When four numbers are proportionals the product of the extremes is equal to the product of the means.*

Let a, b, c, d be proportionals;

then
$$\frac{a}{b} = \frac{c}{d};$$

multiply by bd; thus $ad = bc$.

If any three terms in a proportion are given, the fourth may be determined from the relation $ad = bc$.

If $b = c$ we have $ad = b^2$; that is, *if the first be to the second as the second is to the third, the product of the extremes is equal to the square of the mean.*

When $a : b :: b : d$ then a, b, d are said to be in *continued proportion;* and b is called the *mean proportional* between a and d.

360. *If the product of two numbers be equal to the product of two others, the four are proportionals, the terms of either product being taken for the means, and the terms of the other product for the extremes.*

For let $xy = ab$; divide by ay, thus $\frac{x}{a} = \frac{b}{y}$;

or $x : a :: b : y$ (Art. 357).

361. If $a : b :: c : d$, and $c : d :: e : f$, then $a : b :: e : f$.

For $\frac{a}{b} = \frac{c}{d}$, and $\frac{c}{d} = \frac{e}{f}$; therefore $\frac{a}{b} = \frac{e}{f}$;

or $a : b :: e : f$.

362. *If four numbers be proportionals, they are proportionals when taken inversely;* that is, if $a : b :: c : d$, then $b : a :: d : c$.

For $\frac{a}{b} = \frac{c}{d}$; divide unity by each of these equals; thus $\frac{b}{a} = \frac{d}{c}$; or $b : a :: d : c$.

363. If four numbers be proportionals, they are proportionals when taken alternately; that is, if $a : b :: c : d$, then $a : c :: b : d$.

For $\frac{a}{b} = \frac{c}{d}$; multiply by $\frac{b}{c}$; thus $\frac{a}{c} = \frac{b}{d}$;

or $a : c :: b : d$.

364. *If four numbers are proportionals, the first together with the second is to the second as the third together with the fourth is to the fourth;* that is if $a : b :: c : d$, then $a + b : b :: c + d : d$.

For $\frac{a}{b} = \frac{c}{d}$; add unity to these equals; thus $\frac{a}{b} + 1 = \frac{c}{d} + 1$, that is $\frac{a+b}{b} = \frac{c+d}{d}$; or $a + b : b :: c + d : d$.

365. *Also the excess of the first above the second is to the second as the excess of the third above the fourth is to the fourth.*

For $\frac{a}{b} = \frac{c}{d}$; subtract unity from these equals; thus $\frac{a}{b} - 1 = \frac{c}{d} - 1$, that is $\frac{a-b}{b} = \frac{c-d}{d}$ or $a - b : b :: c - d : d$.

366. *Also the first is to the excess of the first above the second as the third is to the excess of the third above the fourth.*

By the last Article $\frac{a-b}{b}=\frac{c-d}{d}$; also $\frac{a}{b}=\frac{c}{d}$;

therefore $\frac{a-b}{b}\times\frac{b}{a}=\frac{c-d}{d}\times\frac{d}{c}$, or $\frac{a-b}{a}=\frac{c-d}{c}$,

or $a-b : a :: c-d : c$; therefore $a : a-b :: c : c-d$.

367. *When four numbers are proportionals, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference;* that is, if $a : b :: c : d$, then $a+b : a-b :: c+d : c-d$.

By Arts. 364 and 365 $\frac{a+b}{b}=\frac{c+d}{d}$, and $\frac{a-b}{b}=\frac{c-d}{d}$;

therefore $\frac{a+b}{b}\div\frac{a-b}{b}=\frac{c+d}{d}\div\frac{c-d}{d}$, that is $\frac{a+b}{a-b}=\frac{c+d}{c-d}$,

$$a+b : a-b :: c+d : c-d.$$

368. It is obvious from the preceding Articles that if four numbers are proportionals we can derive from them many other proportions; see also Art. 356.

369. In the definition of Proportion it is supposed that we can determine what multiple or what part one quantity is of another quantity of the same kind. But we cannot always do this exactly. For example, if the side of a square is one inch long the length of the diagonal is denoted by $\sqrt{2}$ inches; but $\sqrt{2}$ cannot be exactly found, so that the ratio of the length of the diagonal of a square to the length of a side cannot be exactly expressed by numbers. Two quantities are called *incommensurable* when the ratio of one to the other cannot be exactly expressed by numbers.

The student's acquaintance with Arithmetic will suggest to him that if two quantities are really incommensurable still we may be able to express the ratio of one to the other by numbers as nearly as we please. For example, we can find two mixed numbers, one less than $\sqrt{2}$, and the other greater than $\sqrt{2}$, and one differing from the other by as small a fraction as we please.

370. We will give one proposition with respect to the comparison of two incommensurable quantities.

Let x and y denote two quantities; and suppose it known that however great an integer q may be we can find another integer p such that both x and y lie between $\frac{p}{q}$ and $\frac{p+1}{q}$: then x and y are equal.

For the difference between x and y cannot be so great as $\frac{1}{q}$; and by taking q large enough $\frac{1}{q}$ can be made less than any assigned quantity whatever. But if x and y were unequal their difference could not be made less than any assigned quantity whatever. Therefore x and y must be equal.

371. It will be useful to compare the definition of proportion which has been used in this Chapter with that which is given in the fifth book of Euclid. Euclid's definition may be stated thus: four quantities are proportionals when if any equimultiples be taken of the first and the third, and also any equimultiples of the second and the fourth, the multiple of the third is greater than, equal to, or less than, the multiple of the fourth, according as the multiple of the first is greater than, equal to, or less than the multiple of the second.

372. We will first shew that if four quantities satisfy the algebraical definition of proportion, they will also satisfy Euclid's.

For suppose that $a : b :: c : d$; then $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{pa}{qb} = \frac{pc}{qd}$, whatever numbers p and q may be. Hence pc is greater than, equal to, or less than qd, according as pa is greater than, equal to, or less than qb. That is, the four quantities a, b, c, d satisfy Euclid's definition of proportion.

373. We shall next shew that if four quantities satisfy Euclid's definition of proportion they will also satisfy the algebraical definition.

For suppose that a, b, c, d are four quantities such that whatever numbers p and q may be, pc is greater than,

equal to, or less than qd, according as pa is greater than, equal to, or less than qb.

First suppose that c and d are *commensurable;* take p and q such that $pc = qd$; then by hypothesis $pa = qb$: thus $\frac{pa}{qb} = 1 = \frac{pc}{qd}$; therefore $\frac{a}{b} = \frac{c}{d}$. Therefore $a : b :: c : d$.

Next suppose that c and d are *incommensurable*. Then we cannot find whole numbers p and q, such that $pc = qd$. But we may take any multiple whatever of d, as qd, and this will lie between two consecutive multiples of c, say between pc and $(p+1)c$. Thus $\frac{pc}{qd}$ is less than unity, and $\frac{(p+1)c}{qd}$ is greater than unity. Hence, by hypothesis, $\frac{pa}{qb}$ is less than unity, and $\frac{(p+1)a}{qb}$ is greater than unity. Thus $\frac{c}{d}$ and $\frac{a}{b}$ are both greater than $\frac{p}{q}$, and both less than $\frac{p+1}{q}$. And since this is true however great p and q may be, we infer that $\frac{a}{b}$ and $\frac{c}{d}$ cannot be unequal; that is, they must be equal: see Art. 370. Therefore $a : b :: c : d$.

That is, the four quantities a, b, c, d satisfy the algebraical definition of proportion.

374. It is usually stated that the Algebraical definition of proportion cannot be used in Geometry because there is no method of representing geometrically the result of the operation of division. Straight lines can be represented geometrically, but not the abstract number which expresses how often one straight line is contained in another. But it should be observed that Euclid's definition is rigorous and applicable to *incommensurable* as well as to *commensurable* quantities; while the Algebraical definition is, strictly speaking, confined to the latter. Hence this consideration alone would furnish a sufficient reason for the definition adopted by Euclid.

EXAMPLES. XXXVI.

Find the value of x in each of the following proportions.

1. $4 : 7 :: 8 : x$. 2. $3 : 7 :: x : 42$.

3. $5 : x :: x : 45$. 4. $x : 9 :: 16 : x$.

5. $x+4 : x+2 :: x+8 : x+5$.

6. $x+4 : 2x+8 :: 2x-1 : 3x+2$.

7. $3x+2 : x+7 :: 9x-2 : 5x+8$.

8. $x^2+x+1 : 62(x+1) :: x^2-x+1 : 63(x-1)$.

9. $ax+b : bx+a :: mx+n : nx+m$.

10. If $pq=rs$, and $qt=su$, then $p : r :: t : u$.

11. If $a : b :: c : d$, and $a' : b' :: c' : d'$, then $aa' : bb' :: cc' : dd'$ and $ab' : a'b :: cd' : c'd$.

12. If $a : b :: b : c$, then $(a^2+b^2)(b^2+c^2)=(ab+bc)^2$.

13. There are three numbers in continued proportion; the middle number is 60, and the sum of the others is 125: find the numbers.

14. Find three numbers in continued proportion, such that their sum may be 19, and the sum of their squares 133.

If $a : b :: c : d$, shew that the following relations are true.

15. $a(c+d)=c(a+b)$. 16. $a\sqrt{(c^2+d^2)}=c\sqrt{(a^2+b^2)}$.

17. $$\frac{(a+c)(a^2+c^2)}{(a-c)(a^2-c^2)}=\frac{(b+d)(b^2+d^2)}{(b-d)(b^2-d^2)}.$$

18. $$\frac{pa^2+qab+rb^2}{la^2+mab+nb^2}=\frac{pc^2+qcd+rd^2}{lc^2+mcd+nd^2}.$$

19. $$\frac{1}{a}-\frac{1}{2b}-\frac{1}{3c}+\frac{1}{4d}=\frac{1}{ad}\left\{\frac{a}{4}-\frac{b}{3}-\frac{c}{2}+d\right\}.$$

20. $a : b :: \sqrt[p]{(ma^p+nc^p)} : \sqrt[p]{(mb^p+nd^p)}$.

XXXVII. *Variation.*

375. The present Chapter consists of a series of propositions connected with the definitions of ratio and proportion stated in a new phraseology which is convenient for some purposes.

376. One quantity is said to *vary directly* as another when the two quantities depend on each other, and in such a manner that if one be changed the other is changed in the same proportion.

Sometimes for shortness we omit the word *directly*, and say simply that one quantity varies as another.

377. Thus, for example, if the altitude of a triangle be invariable, the area varies as the base; for if the base be increased or diminished, we know from Euclid that the area is increased or diminished in the same proportion. We may express this result with Algebraical symbols thus; let A and a be *numbers* which represent the areas of two triangles having a common altitude, and let B and b be *numbers* which represent the bases of these triangles respectively; then $\frac{A}{a} = \frac{B}{b}$. And from this we deduce $\frac{A}{B} = \frac{a}{b}$, by Art. 363. If there be a third triangle having the same altitude as the two already considered, then the ratio of the number which represents its area to the number which represents its base will also be equal to $\frac{a}{b}$. Put $\frac{a}{b} = m$, then $\frac{A}{B} = m$, and $A = mB$. Here A may represent the area of *any* one of a series of triangles which have a common altitude, and B the corresponding base, and m remains constant. Hence the statement that the area varies as the base may also be expressed thus, the area has a

constant ratio to the base; by which we mean that the *number* which represents the area bears a constant ratio to the *number* which represents the base.

These remarks are intended to explain the notation and phraseology which are used in the present Chapter. When we say that A varies as B, we mean that A represents the numerical value of any one of a certain series of quantities, and B the numerical value of the corresponding quantity in a certain other series, and that $A = mB$, where m is some number which remains constant for every corresponding pair of quantities.

It will be convenient to give a formal demonstration of the relation $A = mB$, deduced from the definition in Art. 376.

378. *If* A *vary as* B, *then* A *is equal to* B *multiplied by some constant number.*

Let a and b denote one pair of corresponding values of the two quantities, and let A and B denote any other pair; then $\frac{A}{a} = \frac{B}{b}$, by definition. Hence $A = \frac{a}{b} B = mB$, where m is equal to the constant $\frac{a}{b}$.

379. The symbol $\propto$ is used to express variation; thus $A \propto B$ stands for A *varies as* B.

380. One quantity is said to vary *inversely* as another, when the first varies as the *reciprocal* of the second. See Art. 323.

Or if $A = \frac{m}{B}$, where m is constant, A is said to vary inversely as B.

381. One quantity is said to vary as two others *jointly*, when, if the former is changed in any manner, the product of the other two is changed in the same proportion.

Or if $A = mBC$, where m is constant, A is said to vary jointly as B and C.

382. One quantity is said to vary *directly* as a second and *inversely* as a third, when it varies jointly as the second and the reciprocal of the third.

Or if $A = \frac{mB}{C}$, where m is constant, A is said to vary directly as B and inversely as C.

383. *If* $A \propto B$, *and* $B \propto C$, *then* $A \propto C$.

For let $A = mB$, and $B = nC$, where m and n are constants; then $A = mnC$; and, as mn is constant, $A \propto C$.

384. *If* $A \propto C$, *and* $B \propto C$, *then* $A \pm B \propto C$, *and* $\sqrt{(AB)} \propto C$.

For let $A = mC$, and $B = nC$, where m and n are constants; then $A \pm B = (m \pm n)C$; therefore $A \pm B \propto C$.

Also $\sqrt{(AB)} = \sqrt{(mnC^2)} = C\sqrt{(mn)}$; therefore $\sqrt{(AB)} \propto C$.

385. *If* $A \propto BC$, *then* $B \propto \frac{A}{C}$, *and* $C \propto \frac{A}{B}$.

For let $A = mBC$, then $B = \frac{1}{m} \frac{A}{C}$; therefore $B \propto \frac{A}{C}$.

Similarly, $C \propto \frac{A}{B}$.

386. *If* $A \propto B$, *and* $C \propto D$, *then* $AC \propto BD$.

For let $A = mB$, and $C = nD$; then $AC = mnBD$; therefore $AC \propto BD$.

Similarly, if $A \propto B$, and $C \propto D$, and $E \propto F$, then $ACE \propto BDF$; and so on.

387. *If* $A \propto B$, *then* $A^n \propto B^n$.

For let $A = mB$, then $A^n = m^n B^n$; therefore $A^n \propto B^n$.

388. *If* A ∝ B, *then* AP ∝ BP, *where* P *is any quantity variable or invariable.*

For let $A = mB$, then $AP = mBP$; therefore $AP \propto BP$.

389. *If* A ∝ B *when* C *is invariable, and* A ∝ C *when* B *is invariable, then* A ∝ BC *when both* B *and* C *are variable.*

The variation of A depends on the variations of the two quantities B and C; let the variations of the latter quantities take place separately. When B is changed to b let A be changed to a'; then, by supposition, $\frac{A}{a'} = \frac{B}{b}$. Now let C be changed to c, and in consequence let a' be changed to a; then, by supposition, $\frac{a'}{a} = \frac{C}{c}$. Therefore $\frac{A}{a'} \times \frac{a'}{a} = \frac{B}{b} \times \frac{C}{c}$; that is, $\frac{A}{a} = \frac{BC}{bc}$; therefore $A \propto BC$.

A very good example of this proposition is furnished in Geometry. It can be shewn that the area of a triangle varies as the base when the height is invariable, and that the area varies as the height when the base is invariable. Hence when both the base and the height vary, the area varies as the product of the numbers which represent the base and the height.

Other examples of this proposition are supplied by the questions which occur in Arithmetic under the head of the Double Rule of Three. For instance suppose that the quantity of a work which can be accomplished varies as the number of workmen when the time is given, and varies as the time when the number of workmen is given; then the quantity of the work will vary as the product of the number of workmen and the time when both vary.

390. In the same manner, if there be any number of quantities B, C, D, ... each of which varies as another quantity A when the rest are constant, when they all vary A varies as their product.

EXAMPLES. XXXVII.

1. A varies as B, and $A=2$ when $B=1$; find the value of A when $B=2$.

2. If A^2+B^2 varies as A^2-B^2, shew that $A+B$ varies as $A-B$.

3. $3A+5B$ varies as $5A+3B$, and $A=5$ when $B=2$; find the ratio $A:B$.

4. A varies as $nB+C$; and $A=4$ when $B=1$, and $C=2$; and $A=7$ when $B=2$, and $C=3$: find n.

5. A varies as B and C jointly; and $A=1$ when $B=1$, and $C=1$: find the value of A when $B=2$ and $C=2$.

6. A varies as B and C jointly; and $A=8$ when $B=2$, and $C=2$: find the value of BC when $A=10$.

7. A varies as B and C jointly; and $A=12$ when $B=2$, and $C=3$: find the value of $A:B$ when $C=4$.

8. A varies as B and C jointly; and $A=a$ when $B=b$, and $C=c$: find the value of A when $B=b^2$ and $C=c^2$.

9. A varies as B directly and as C inversely; and $A=a$ when $B=b$, and $C=c$: find the value of A when $B=c$ and $C=b$.

10. The expenses of a Charitable Institution are partly constant, and partly vary as the number of inmates. When the inmates are 960 and 3000 the expenses are respectively £112 and £180. Find the expenses for 1000 inmates.

11. The wages of 5 men for 7 weeks being £17. 10*s*. find how many men can be hired to work 4 weeks for £30.

12. If the cost of making an embankment vary as the length if the area of the transverse section and height be constant, as the height if the area of the transverse section and length be constant, and as the area of the transverse section if the length and height be constant, and an embankment 1 mile long, 10 feet high, and 12 feet broad cost £9600 find the cost of an embankment half a mile long, 16 feet high, and 15 feet broad.

XXXVIII. *Arithmetical Progression.*

391. Quantities are said to be in Arithmetical Progression when they increase or decrease by a common difference.

Thus the following series are in Arithmetical Progression,

$$2, 5, 8, 11, 14, \ldots\ldots$$
$$20, 18, 16, 14, 12, \ldots\ldots$$
$$a, a+b, a+2b, a+3b, a+4b \ldots\ldots$$

The common difference is found by subtracting any term from that which immediately follows it. In the first series the common difference is 3; in the second series it is -2; in the third series it is b.

392. Let a denote the first term of an Arithmetical Progression, b the common difference; then the second term is $a+b$, the third term is $a+2b$, the fourth term is $a+3b$, and so on. Thus the n^{th} term is $a+(n-1)b$.

393. *To find the sum of a given number of terms of an Arithmetical Progression, the first term and the common difference being supposed known.*

Let a denote the first term, b the common difference, n the number of terms, l the last term, s the sum of the terms. Then

$$s = a + (a+b) + (a+2b) + \ldots\ldots + l.$$

And, by writing the series in the reverse order, we have also

$$s = l + (l-b) + (l-2b) + \ldots\ldots + a.$$

Therefore, by addition,

$$2s = (l+a) + (l+a) + \ldots\ldots \text{to } n \text{ terms}$$
$$= n(l+a);$$

therefore $$s = \frac{n}{2}(l+a) \ldots\ldots\ldots\ldots\ldots (1).$$

Also $$l = a + (n-1)b \ldots\ldots\ldots\ldots(2),$$

thus $$s = \frac{n}{2}\{2a + (n-1)b\}\ldots\ldots(3).$$

The equation (3) gives the value of s in terms of the quantities which were supposed known. Equation (1) also gives a convenient expression for s, and furnishes the following rule: the *sum of any number of terms in Arithmetical Progression is equal to the product of the number of the terms into half the sum of the first and last terms.*

We shall now apply the equations in the present Article to solve some examples relating to Arithmetical Progression.

394. Find the sum of 20 terms of the series 1, 2, 3, 4,...

Here $a = 1$, $b = 1$, $n = 20$; therefore

$$s = \frac{20}{2}(2 + 19) = 10 \times 21 = 210.$$

395. Find the sum of 20 terms of the series, 1, 3, 5, 7,...

Here $a = 1$, $b = 2$, $n = 20$; therefore,

$$s = \frac{20}{2}(2 + 19 \times 2) = \frac{20}{2} \times 40 = (20)^2 = 400.$$

396. Find the sum of 12 terms of the series 20, 18, 16,...

Here $a = 20$, $b = -2$, $n = 12$; therefore

$$s = \frac{12}{2}(40 - 2 \times 11) = 6(40 - 22) = 6 \times 18 = 108.$$

397. Find the sum of 8 terms of the series $\frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \ldots$

Here $a = \frac{1}{12}$, $b = \frac{1}{12}$, $n = 8$; therefore

$$s = \frac{8}{2}\left(\frac{2}{12} + \frac{7}{12}\right) = 4 \times \frac{9}{12} = 3.$$

398. How many terms must be taken of the series 15, 12, 9,... that the sum may be 42?

Here $s=42$, $a=15$, $b=-3$; therefore

$$42=\frac{n}{2}\left\{30-3(n-1)\right\}=\frac{n}{2}(33-3n).$$

We have to find n from this quadratic equation; by solving it we shall obtain $n=4$ or 7. The series is 15, 12, 9, 6, 3, 0, −3,......; and thus it will be found that we obtain 42 as the sum of the first 4 terms, or as the sum of the first 7 terms.

399. Insert five Arithmetical means between 11 and 23.

Here we have to obtain an Arithmetical Progression consisting of *seven* terms, beginning with 11 and ending with 23. Thus $a=11$, $l=23$, $n=7$; therefore by equation (2) of Art. 393,

$$23=11+6b,$$

$$\text{therefore } b=2.$$

Thus the whole series is 11, 13, 15, 17, 19, 21, 23.

EXAMPLES. XXXVIII.

Sum the following series:

1. 100, 101, 102,.........to 9 terms.
2. 1, $2\frac{1}{2}$, 4, ...to 10 terms.
3. 1, $2\frac{2}{3}$, $4\frac{1}{3}$,................to 9 terms.
4. 2, $3\frac{3}{4}$, $5\frac{1}{2}$,................to 12 terms.
5. $\frac{2}{3}$, $\frac{5}{6}$, 1,................to 18 terms.
6. $\frac{1}{2}$, $-\frac{2}{3}$, $-\frac{11}{6}$,.........to 15 terms.
7. Insert 3 Arithmetical means between 12 and 20.
8. Insert 5 Arithmetical means between 14 and 16.

9. Insert 7 Arithmetical means between 8 and -4.

10. Insert 8 Arithmetical means between -1 and 5.

11. The first term of an Arithmetical Progression is 13, the second term is 11, the sum is 40: find the number of terms.

12. The first term of an Arithmetical Progression is 5, and the fifth term is 11: find the sum of 8 terms.

13. The sum of four terms in Arithmetical Progression is 44, and the last term is 17: find the terms.

14. The sum of three numbers in Arithmetical Progression is 21, and the sum of their squares is 155: find the numbers.

15. The sum of five numbers in Arithmetical Progression is 15, and the sum of their squares is 55: find the numbers.

16. The seventh term of an Arithmetical Progression is 12, and the twelfth term is 7; the sum of the series is 171: find the number of terms.

17. A traveller has a journey of 140 miles to perform. He goes 26 miles the first day, 24 the second, 22 the third, and so on. In how many days does he perform the journey?

18. A sets out from a place and travels $2\frac{1}{2}$ miles an hour. B sets out 3 hours after A, and travels in the same direction, 3 miles the first hour, $3\frac{1}{2}$ miles the second, 4 miles the third, and so on. In how many hours will B overtake A?

19. The sum of three numbers in Arithmetical Progression is 12; and the sum of their squares is 66: find the numbers.

20. If the sum of n terms of an Arithmetical Progression is always equal to n^2, find the first term and the common difference.

XXXIX. *Geometrical Progression.*

400. Quantities are said to be in Geometrical Progression when each is equal to the product of the preceding and some constant factor. The constant factor is called the *common ratio* of the series, or more shortly, the *ratio.*

Thus the following series are in Geometrical Progression.

$$1, 3, 9, 27, 81, \ldots\ldots$$

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\ldots$$

$$a, ar, ar^2, ar^3, ar^4, \ldots\ldots$$

The common ratio is found by dividing any term by that which immediately precedes it. In the first example the common ratio is 3, in the second it is $\frac{1}{2}$, in the third it is r.

401. Let a denote the first term of a Geometrical Progression, r the common ratio; then the second term is ar, the third term is ar^2, the fourth term is ar^3, and so on. Thus the n^{th} term is ar^{n-1}.

402. *To find the sum of a given number of terms of a Geometrical Progression, the first term and the common ratio being supposed known.*

Let a denote the first term, r the common ratio, n the number of terms, s the sum of the terms. Then

$$s = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1};$$

therefore $sr = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n.$

Therefore, by subtraction,

$$sr - s = ar^n - a,$$

therefore $$s = \frac{a(r^n - 1)}{r - 1} \ldots\ldots\ldots\ldots\ldots(1).$$

If l denote the last term we have

$$l = ar^{n-1} \quad \text{..................(2)},$$

therefore
$$s = \frac{rl - a}{r - 1} \quad \text{..................(3)}.$$

Equation (1) gives the value of s in terms of the quantities which were supposed known. Equation (3) is sometimes a convenient form.

We shall now apply these equations to solve some examples relating to Geometrical Progression.

403. Find the sum of 6 terms of the series 1, 3, 9, 27,...

Here $a=1$, $r=3$, $n=6$; therefore

$$s = \frac{3^6 - 1}{3 - 1} = \frac{729 - 1}{3 - 1} = 364.$$

404. Find the sum of 6 terms of the series 1, -3, 9, -27,...

Here $a=1$, $r=-3$, $n=6$; therefore

$$s = \frac{(-3)^6 - 1}{-3 - 1} = \frac{729 - 1}{-4} = -182.$$

405. Find the sum of 8 terms of the series 4, 2, 1, $\frac{1}{2}$,...

Here $a=4$, $r=\frac{1}{2}$, $n=8$; therefore

$$s = \frac{4\left(\frac{1}{2^8} - 1\right)}{\frac{1}{2} - 1} = \frac{4\left(1 - \frac{1}{2^8}\right)}{1 - \frac{1}{2}} = \frac{255}{64} \times \frac{2}{1} = \frac{255}{32}.$$

406. Find the sum of 7 terms of the series, 8, -4, 2, -1, $\frac{1}{2}$, ...

Here $a=8$, $r=-\frac{1}{2}$, $n=7$; therefore

$$s = \frac{8\left\{\left(-\frac{1}{2}\right)^7 - 1\right\}}{-\frac{1}{2} - 1} = \frac{8\left(-\frac{1}{128} - 1\right)}{-\frac{1}{2} - 1} = \frac{129}{16} \times \frac{2}{3} = \frac{43}{8}.$$

407. Insert three Geometrical means between 2 and 32.

Here we have to obtain a Geometrical Progression consisting of *five* terms, beginning with 2 and ending with 32. Thus $a=2$, $l=32$, $n=5$; therefore, by equation (2) of Art. 402,

$$32=2r^4,$$

that is $$r^4=16=2^4;$$

therefore $$r=2.$$

Thus the whole series is 2, 4, 8, 16, 32.

408. We may write the value of s, given in Art. 402, thus

$$s=\frac{a(1-r^n)}{1-r}.$$

Now suppose that r is *less than unity;* then the larger n is, the smaller will r^n be, and by taking n large enough r^n can be made as small as we please. If we neglect r^n we obtain

$$s=\frac{a}{1-r},$$

and we may enunciate the result thus. *In a Geometrical Progression in which the common ratio is numerically less than unity, by taking a sufficient number of terms the sum can be made to differ as little as we please from* $\frac{a}{1-r}$.

409. For example, take the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

Here $a=1$, $r=\frac{1}{2}$; therefore $\frac{a}{1-r}=2$. Thus by taking a sufficient number of terms the sum can be made to differ as little as we please from 2. In fact if we take *four* terms the sum is $2-\frac{1}{8}$, if we take *five* terms the sum is $2-\frac{1}{16}$, if we take *six* terms the sum is $2-\frac{1}{32}$, and so on.

The result is sometimes expressed thus for shortness, *the sum of an infinite number of terms of this series is* 2; or thus, *the sum to infinity is* 2.

410. Recurring decimals are examples of what are called infinite Geometrical Progression. Thus for example $\cdot 3242424...$ denotes $\frac{3}{10}+\frac{24}{10^3}+\frac{24}{10^5}+\frac{24}{10^7}+...$

Here the terms after $\frac{3}{10}$ form a Geometrical Progression, of which the first term is $\frac{24}{10^3}$, and the common ratio is $\frac{1}{10^2}$. Hence we may say that the sum of an infinite number of terms of this series is $\frac{24}{10^3}\div\left(1-\frac{1}{10^2}\right)$, that is $\frac{24}{990}$. Therefore the value of the recurring decimal is $\frac{3}{10}+\frac{24}{990}$.

The value of the recurring decimal may be found practically thus:

Let $s = \cdot 32424...;$
then $10s = 3\cdot 2424...,$
and $1000s = 324\cdot 2424...$

Hence, by subtraction, $(1000-10)s = 324-3 = 321;$
therefore $s = \frac{321}{990}$.

And any other example may be treated in a similar manner.

EXAMPLES. XXXIX.

Sum the following series:

1. 1, 4, 16, ... to 6 terms.
2. 9, 3, 1, ... to 5 terms.
3. 25, 10, 4, ... to 4 terms.
4. 1, $\sqrt{2}$, 2, $2\sqrt{2}$, ... to 12 terms.

5. $\frac{3}{8}, \frac{1}{4}, \frac{1}{6}, \ldots\ldots\ldots$ to 6 terms.

6. $\frac{2}{3}, -1, \frac{3}{2}, \ldots\ldots\ldots$ to 7 terms.

7. $1, -\frac{1}{3}, \frac{1}{9}, \ldots\ldots$ to infinity.

8. $1, \frac{1}{4}, \frac{1}{16}, \ldots\ldots\ldots$ to infinity.

9. $1, -\frac{1}{2}, \frac{1}{4}, \ldots\ldots$ to infinity.

10. $6, -2, \frac{2}{3}, \ldots\ldots\ldots$ to infinity.

Find the value of the following recurring decimals:

11. ·151515... 12. ·123123123...

13. ·4282828... 14. ·28131313...

15. Insert 3 Geometrical means between 1 and 256.

16. Insert 4 Geometrical means between $5\frac{1}{3}$ and $40\frac{1}{2}$.

17. Insert 4 Geometrical means between 3 and -729.

18. The sum of three terms in Geometrical Progression is 63, and the difference of the first and third terms is 45: find the terms.

19. The sum of the first four terms of a Geometrical Progression is 40, and the sum of the first eight terms is 3280: find the Progression.

20. The sum of three terms in Geometrical Progression is 21, and the sum of their squares is 189: find the terms.

XL. *Harmonical Progression.*

411. Three quantities A, B, C are said to be in Harmonical Progression when $A : C :: A-B : B-C$.

Any number of quantities are said to be in Harmonical Progression when every three consecutive quantities are in Harmonical Progression.

412. *The reciprocals of quantities in Harmonical Progression are in Arithmetical Progression.*

Let A, B, C be in Harmonical Progression; then $A : C :: A-B : B-C$.

Therefore $A(B-C) = C(A-B)$.

Divide by ABC; thus $\frac{1}{C} - \frac{1}{B} = \frac{1}{B} - \frac{1}{A}$.

This demonstrates the proposition.

413. The property established in the preceding Article will enable us to solve some questions relating to Harmonical Progression. For example, insert five Harmonical means between $\frac{2}{3}$ and $\frac{8}{15}$. Here we have to insert five *Arithmetical* means between $\frac{3}{2}$ and $\frac{15}{8}$. Hence, by equation (2) of Art. 393,

$$\frac{15}{8} = \frac{3}{2} + 6b,$$

therefore $6b = \frac{3}{8}$, therefore $b = \frac{1}{16}$.

Hence the Arithmetical Progression is $\frac{3}{2}$, $\frac{25}{16}$, $\frac{26}{16}$, $\frac{27}{16}$, $\frac{28}{16}$, $\frac{29}{16}$, $\frac{15}{8}$; and therefore the Harmonical Progression is $\frac{2}{3}$, $\frac{16}{25}$, $\frac{16}{26}$, $\frac{16}{27}$, $\frac{16}{28}$, $\frac{16}{29}$, $\frac{8}{15}$.

414. Let a and c be any two quantities; let A be their Arithmetical mean, G their Geometrical mean, H their Harmonical mean. Then

$$A-a=c-A\text{ ; therefore } A=\frac{1}{2}(a+c).$$

$$a : G :: G : c\text{; therefore } G=\sqrt{(ac)}.$$

$$a : c :: a-H : H-c\text{; therefore } H=\frac{2ac}{a+c}.$$

EXAMPLES. XL.

1. Continue the Harmonical Progression 6, 3, 2 for three terms.

2. Continue the Harmonical Progression 8, 2, $1\frac{1}{7}$ for three terms.

3. Insert 2 Harmonical means between 4 and 2.

4. Insert 3 Harmonical means between $\frac{1}{3}$ and $\frac{1}{21}$.

5. The Arithmetical mean of two numbers is 9, and the Harmonical mean is 8: find the numbers.

6. The Geometrical mean of two numbers is 48, and the Harmonical mean is $46\frac{2}{25}$: find the numbers.

7. Find two numbers such that the sum of their Arithmetical, Geometrical, and Harmonical means is $9\frac{4}{5}$, and the product of these means is 27.

8. Find two numbers such that the product of their Arithmetical and Harmonical means is 27, and the excess of the Arithmetical mean above the Harmonical mean is $1\frac{1}{2}$.

9. If a, b, c are in Harmonical Progression, shew that

$$a+c-2b : a-c :: a-c : a+c.$$

10. If three numbers are in Geometrical Progression, and each of them is increased by the middle number, shew that the results are in Harmonical Progression.

XLI. *Permutations and Combinations.*

415. The different orders in which a set of things can be arranged are called their *permutations.*

Thus the permutations of the three letters a, b, c, taken two at a time, are ab, ba, ac, ca, bc, cb.

416. The *combinations* of a set of things are the different collections which can be formed out of them, without regarding the order in which the things are placed.

Thus the combinations of the three letters a, b, c, taken two at a time, are ab, ac, bc; ab and ba, though different *permutations*, form the same *combination*, so also do ac and ca, and bc and cb.

417. *The number of permutations of* n *things taken* r *at a time is* $n(n-1)(n-2)\ldots\ldots(n-r+1)$.

Let there be n letters a, b, c, d, ...; we shall first find the number of permutations of them taken *two* at a time. Put a before each of the other letters; we thus obtain $n-1$ permutations in which a stands first. Put b before each of the other letters; we thus obtain $n-1$ permutations in which b stands first. Similarly there are $n-1$ permutations in which c stands first. And so on. Thus, on the whole, there are $n(n-1)$ permutations of n letters taken *two* at a time. We shall next find the number of permutations of n letters taken *three* at a time. It has just been shewn that out of n letters we can form $n(n-1)$ permutations, each of two letters; hence out of the $n-1$ letters b, c, d, ... we can form $(n-1)(n-2)$ permutations, each of two letters: put a before each of these, and we have $(n-1)(n-2)$ permutations, each of three letters, in which a stands first. Similarly there are $(n-1)(n-2)$ permutations, each of three letters, in which b stands first. Similarly there are as many in which c stands first. And so on. Thus, on the whole, there are $n(n-1)(n-2)$ permutations of n letters taken *three* at a time.

From considering these cases it might be conjectured that the number of permutations of n letters taken r at a time is $n(n-1)(n-2)\ldots(n-r+1)$; and we shall shew that this is the case. For suppose it known that the number of permutations of n letters taken $r-1$ at a time is $n(n-1)(n-2)\ldots\{n-(r-1)+1\}$, we shall shew that a similar formula will give the number of permutations of n letters, taken r at a time. For out of the $n-1$ letters b, c, d,... we can form $(n-1)(n-2)\ldots\ldots\{n-1-(r-1)+1\}$ permutations, each of $r-1$ letters: put a before each of these, and we obtain as many permutations, each of r letters, in which a stands first. Similarly there are as many permutations, each of r letters, in which b stands first. Similarly there are as many permutations, each of r letters, in which c stands first. And so on. Thus on the whole there are $n(n-1)(n-2)\ldots(n-r+1)$ permutations of n letters taken r at a time.

If then the formula holds when the letters are taken $r-1$ at a time it will hold when they are taken r at a time. But it has been shewn to hold when they are taken *three* at a time, therefore it holds when they are taken *four* at a time, and therefore it holds when they are taken *five* at a time, and so on: thus it holds universally.

418. Hence the number of permutations of n things taken all together is $n(n-1)(n-2)\ldots 1$.

419. For the sake of brevity $n(n-1)(n-2)\ldots 1$ is often denoted by $\underline{|n}$; thus $\underline{|n}$ denotes the product of the natural numbers from 1 to n inclusive. The symbol $\underline{|n}$ may be read, *factorial n*.

420. *Any combination of* r *things will produce* $\underline{|r}$ *permutations.*

For by Art. 418 the r things which form the given combination can be arranged in $\underline{|r}$ different orders.

421. *The number of combinations of* n *things taken* r *at a time is* $\dfrac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}}$.

For the number of *permutations* of n things taken r at a time is $n(n-1)(n-2)\dots(n-r+1)$ by Art. 417; and each combination produces $\lfloor r$ permutations by Art. 420; hence the number of combinations must be

$$\frac{n(n-1)(n-2)\dots(n-r+1)}{\lfloor r}.$$

If we multiply both numerator and denominator of this expression by $\lfloor n-r$ it takes the form $\dfrac{\lfloor n}{\lfloor r \lfloor n-r}$, the value of course being unchanged.

422. *To find the number of permutations of* n *things taken all together which are not all different.*

Let there be n letters; and suppose p of them to be a, q of them to be b, r of them to be c, and the rest of them to be the letters d, e, ..., each occurring singly: then the number of permutations of them taken all together will be

$$\frac{\lfloor n}{\lfloor p \lfloor q \lfloor r}.$$

For suppose N to represent the required number of permutations. If in any one of the permutations the p letters a were changed into p new and different letters, then, without changing the situation of any of the other letters, we could from the single permutation produce $\lfloor p$ different permutations: and thus if the p letters a were changed into p new and different letters the whole number of permutations would be $N \times \lfloor p$. Similarly if the q letters b were also changed into q new and different letters the whole number of permutations we could now obtain would be $N \times \lfloor p \times \lfloor q$. And if the r letters c were also changed into r new and different letters the whole number of permutations would be $N \times \lfloor p \times \lfloor q \times \lfloor r$. But this number must be equal to the number of permutations of n different letters taken all together, that is to $\lfloor n$.

Thus $N \times \lfloor p \times \lfloor q \times \lfloor r = \lfloor n$; therefore $N = \dfrac{\lfloor n}{\lfloor p \lfloor q \lfloor r}$.

And similarly any other case may be treated.

423. The student should notice the peculiar method of demonstration which is employed in Art. 417. This is called *mathematical induction*, and may be thus described: We shew that if a theorem is true in *one* case, whatever that case may be, it is also true in another case so related to the former that it may be called the *next* case; we also shew in some manner that the theorem *is* true in a certain case; hence it is true in the next case, and hence in the next to that, and so on; thus finally the theorem must be true in every case after that with which we began.

The method of mathematical induction is frequently used in the higher parts of mathematics.

EXAMPLES. XLI.

1. Find how many parties of 6 men each can be formed from a company of 24 men.

2. Find how many permutations can be formed of the letters in the word *company*, taken all together.

3. Find how many combinations can be formed of the letters in the word *longitude*, taken four at a time.

4. Find how many permutations can be formed of the letters in the word *consonant*, taken all together.

5. The number of the combinations of a set of things taken *four* at a time is twice as great as the number taken *three* at a time: find how many things there are in the set.

6. Find how many words each containing two consonants and one vowel can be formed from 20 consonants and 5 vowels, the vowel being the middle letter of the word.

7. Five persons are to be chosen by lot out of twenty: find in how many ways this can be done. Find also how often an assigned person would be chosen.

8. A boat's crew consisting of eight rowers and a steersman is to be formed out of twelve persons, nine of whom can row but cannot steer, while the other three can steer but cannot row: find in how many ways the crew can be formed. Find also in how many ways the crew could be formed if one of the three were able both to row and to steer.

XLII. *Binomial Theorem.*

424. We have already seen that $(x+a)^2 = x^2 + 2xa + a^2$, and that $(x+a)^3 = x^3 + 3x^2a + 3xa^2 + a^3$; the object of the present Chapter is to find an expression for $(x+a)^n$ where n is any positive integer.

425. By actual multiplication we obtain

$$(x+a)(x+b) = x^2 + (a+b)x + ab,$$

$$(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ab+bc+ca)x + abc,$$

$$\begin{aligned}(x+a)(x+b)(x+c)(x+d) = x^4 &+ (a+b+c+d)x^3 \\ &+ (ab+ac+ad+bc+bd+cd)x^2 \\ &+ (abc+bcd+cda+dab)x + abcd.\end{aligned}$$

Now in these results we see that the following laws hold:

I. The number of terms on the right-hand side is one more than the number of binomial factors which are multiplied together.

II. The exponent of x in the first term is the same as the number of binomial factors, and in the other terms each exponent is less than that of the preceding term by unity.

III. The coefficient of the first term is unity; the coefficient of the second term is the sum of the second letters of the binomial factors; the coefficient of the third term is the sum of the products of the second letters of the binomial factors taken two at a time; the coefficient of the fourth term is the sum of the products of the second letters of the binomial factors taken three at a time; and so on; the last term is the product of all the second letters of the binomial factors.

We shall shew that these laws always hold, whatever be the number of binomial factors. Suppose the laws to hold when $n-1$ factors are multiplied together; that is,

suppose there are $n-1$ factors $x+a$, $x+b$, $x+c$, ... $x+k$, and that

$$(x+a)(x+b)\ldots(x+k)=x^{n-1}+px^{n-2}+qx^{n-3}+rx^{n-4}+\ldots+u,$$

where $p=$ the sum of the letters a, b, c, ... k,

$q=$ the sum of the products of these letters taken two at a time,

$r=$ the sum of the products of these letters taken three at a time,

$u=$ the product of all these letters.

Multiply both sides of this identity by another factor $x+l$, and arrange the product on the right hand according to powers of x; thus

$$(x+a)(x+b)(x+c)\ldots(x+k)(x+l)=x^n+(p+l)x^{n-1}$$
$$+(q+pl)x^{n-2}+(r+ql)x^{n-3}+\ldots+ul.$$

Now $p+l=a+b+c+\ldots+k+l$

$=$ the sum of all the letters a, b, c,...k, l;

$q+pl=q+l(a+b+c+\ldots+k)$

$=$ the sum of the products taken two at a time of all the letters a, b, c,...k, l;

$r+ql=r+l(ab+ac+bc+\ldots)$

$=$ the sum of the products taken three at a time of all the letters a, b, c,...k, l;

$ul=$ the product of all the letters.

Hence, if the laws hold when $n-1$ factors are multiplied together, they hold when n factors are multiplied together; but they have been shewn to hold when *four* factors are multiplied together, therefore they hold when *five* factors are multiplied together, and so on: thus they hold universally.

We shall write the result for the multiplication of n factors thus for abbreviation:

$$(x+a)(x+b)\ldots(x+k)(x+l)=x^n+Px^{n-1}+Qx^{n-2}+Rx^{n-3}+\ldots+V.$$

Now P is the sum of the letters $a, b, c, \ldots k, l$, which are n in number; Q is the sum of the products of these letters two and two, so that there are $\frac{n(n-1)}{1.2}$ of these products; R is the sum of $\frac{n(n-1)(n-2)}{1.2.3}$ products; and so on. See Art. 421.

Suppose $b, c, \ldots k, l$ each equal to a. Then P becomes na, Q becomes $\frac{n(n-1)}{1.2}a^2$, R becomes $\frac{n(n-1)(n-2)}{1.2.3}a^3$; and so on. Thus finally

$$(x+a)^n=x^n+nax^{n-1}+\frac{n(n-1)}{1.2}a^2x^{n-2}+\frac{n(n-1)(n-2)}{1.2.3}a^3x^{n-3}$$
$$+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}a^4x^{n-4}+\ldots\ldots\ldots\ldots+a^n.$$

426. The formula just obtained is called the Binomial Theorem; the series on the right-hand side is called the expansion of $(x+a)^n$, and when we put this series instead of $(x+a)^n$ we are said to *expand* $(x+a)^n$. The theorem was discovered by Newton.

It will be seen that we have demonstrated the theorem in the case in which the exponent n is a *positive integer;* and that we have used in this demonstration the method of *mathematical induction.*

427. Take for example $(x+a)^6$. Here $n=6$,

$$\frac{n(n-1)}{1.2}=\frac{6.5}{1.2}=15,\quad \frac{n(n-1)(n-2)}{1.2.3}=\frac{6.5.4}{1.2.3}=20,$$

$$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}=\frac{6.5.4.3}{1.2.3.4}=15,$$

$$\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}=\frac{6.5.4.3.2}{1.2.3.4.5}=6;$$

thus

$$(x+a)^6 = x^6 + 6ax^5 + 15a^2x^4 + 20a^3x^3 + 15a^4x^2 + 6a^5x + a^6.$$

Again, suppose we require the expansion of $(b^2+cy)^6$: we have only to put b^2 for x and cy for a in the preceding identity; thus

$$(b^2+cy)^6 = (b^2)^6 + 6cy(b^2)^5 + 15(cy)^2(b^2)^4 + 20(cy)^3(b^2)^3$$
$$+ 15(cy)^4(b^2)^2 + 6(cy)^5b^2 + (cy)^6 = b^{12} + 6cyb^{10} + 15c^2y^2b^8$$
$$+ 20c^3y^3b^6 + 15c^4y^4b^4 + 6c^5y^5b^2 + c^6y^6.$$

Again, suppose we require the expansion of $(x-c)^n$; we must put $-c$ for a in the result of Art. 425; thus

$$(x-c)^n = x^n - ncx^{n-1} + \frac{n(n-1)}{1.2}c^2x^{n-2}$$
$$- \frac{n(n-1)(n-2)}{1.2.3}c^3x^{n-3} + \dots$$

Again, in the expansion of $(x+a)^n$ put 1 for x; thus

$$(1+a)^n = 1 + na + \frac{n(n-1)}{1.2}a^2 + \frac{n(n-1)(n-2)}{1.2.3}a^3 + \dots$$

and as this is true for all values of a we may put x for a; thus

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \dots$$

428. We may apply the Binomial Theorem to expand expressions containing more than two terms. For example, required to expand $(1+2x-x^2)^4$. Put y for $2x-x^2$; then we have $(1+2x-x^2)^4 = (1+y)^4 = 1 + 4y + 6y^2 + 4y^3 + y^4$

$$= 1 + 4(2x-x^2) + 6(2x-x^2)^2 + 4(2x-x^2)^3 + (2x-x^2)^4.$$

Also $(2x-x^2)^2 = (2x)^2 - 2(2x)x^2 + (x^2)^2 = 4x^2 - 4x^3 + x^4$,

$$(2x-x^2)^3 = (2x)^3 - 3(2x)^2x^2 + 3(2x)(x^2)^2 - (x^2)^3$$
$$= 8x^3 - 12x^4 + 6x^5 - x^6,$$
$$(2x-x^2)^4 = (2x)^4 - 4(2x)^3x^2 + 6(2x)^2(x^2)^2 - 4(2x)(x^2)^3 + (x^2)^4$$
$$= 16x^4 - 32x^5 + 24x^6 - 8x^7 + x^8.$$

Hence, collecting the terms, we obtain $(1+2x-x^2)^4$

$$=1+8x+20x^2+8x^3-26x^4-8x^5+20x^6-8x^7+x^8.$$

429. *In the expansion of* $(1+x)^n$ *the coefficients of terms equally distant from the beginning and the end are the same.*

The coefficient of the r^{th} term from the beginning is $\dfrac{n(n-1)(n-2)\dots(n-r+2)}{\underline{\lfloor r-1}}$; by multiplying both numerator and denominator by $\underline{\lfloor n-r+1}$ this becomes $\dfrac{\underline{\lfloor n}}{\underline{\lfloor r-1}\,\underline{\lfloor n-r+1}}$.

The r^{th} term from the end is the $(n-r+2)^{th}$ term from the beginning, and its coefficient is

$$\frac{n(n-1)\dots\{n-(n-r+2)+2\}}{\underline{\lfloor n-r+1}}, \text{ that is } \frac{n(n-1)\dots r}{\underline{\lfloor n-r+1}};$$

by multiplying both numerator and denominator by $\underline{\lfloor r-1}$ this also becomes $\dfrac{\underline{\lfloor n}}{\underline{\lfloor r-1}\,\underline{\lfloor n-r+1}}$.

430. Hitherto in speaking of the expansion of $(x+a)^n$ we have assumed that n denotes *some positive integer.* But the Binomial Theorem is also applied to expand $(x+a)^n$ when n is a positive fraction, or a negative quantity whole or fractional. For a discussion of the Binomial Theorem with any exponent the student is referred to the larger Algebra; it will however be a useful exercise to obtain various particular cases from the general formula. Thus the student will assume for the present that whatever be the values of x, a, and n,

$$(x+a)^n=x^n+nax^{n-1}+\frac{n(n-1)}{1.2}a^2x^{n-2}+\frac{n(n-1)(n-2)}{1.2.3}a^3x^{n-3}$$
$$+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}a^3x^{n-3}+\dots\dots$$

If n is not a positive integer the series never ends.

431. As an example take $(1+y)^{\frac{1}{2}}$. Here in the formula of Art. 430 we put 1 for x, y for a, and $\frac{1}{2}$ for n.

$$\frac{n(n-1)}{1.2} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1.2} = -\frac{1}{8},$$

$$\frac{n(n-1)(n-2)}{1.2.3} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1.2.3} = \frac{1}{16},$$

$$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{1.2.3.4} = -\frac{5}{128},$$

and so on. Thus

$$(1+y)^{\frac{1}{2}} = 1 + \frac{1}{2}y - \frac{1}{8}y^2 + \frac{1}{16}y^3 - \frac{5}{128}y^4 + \ldots\ldots$$

As another example take $(1+y)^{-\frac{1}{2}}$. Here we put 1 for x, y for a, and $-\frac{1}{2}$ for n.

$$n = -\frac{1}{2}, \quad \frac{n(n-1)}{1.2} = \frac{3}{8}, \quad \frac{n(n-1)(n-2)}{1.2.3} = -\frac{5}{16},$$

$$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4} = \frac{35}{128}, \text{ and so on. Thus}$$

$$(1+y)^{-\frac{1}{2}} = 1 - \frac{1}{2}y + \frac{3}{8}y^2 - \frac{5}{16}y^3 + \frac{35}{128}y^4 - \ldots\ldots$$

Again, expand $(1+y)^{-m}$. Here we put 1 for x, y for a, and $-m$ for n.

$$n = -m, \quad \frac{n(n-1)}{1.2} = \frac{m(m+1)}{1.2},$$

$$\frac{n(n-1)(n-2)}{1.2.3} = -\frac{m(m+1)(m+2)}{1.2.3},$$

$$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4} = \frac{m(m+1)(m+2)(m+3)}{1.2.3.4}, \text{ and so on.}$$

Thus $(1+y)^{-m} = 1 - my + \frac{m(m+1)}{1.2} y^2 - \frac{m(m+1)(m+2)}{1.2.3} y^3$

$$+ \frac{m(m+1)(m+2)(m+3)}{1.2.3.4} y^4 - \ldots$$

As a particular case suppose $m = 1$; thus

$$(1+y)^{-1} = 1 - y + y^2 - y^3 + y^4 - \ldots$$

This may be verified by dividing 1 by $1+y$.

Again, expand $(1+2x-x^2)^{\frac{1}{2}}$ in powers of x. Put y for $2x - x^2$; thus we have $(1+2x-x^2)^{\frac{1}{2}} = (1+y)^{\frac{1}{2}}$

$$= 1 + \frac{1}{2} y - \frac{1}{8} y^2 + \frac{1}{16} y^3 - \frac{5}{128} y^4 + \ldots$$

$$= 1 + \frac{1}{2}(2x - x^2) - \frac{1}{8}(2x - x^2)^2 + \frac{1}{16}(2x - x^2)^3 - \frac{5}{128}(2x - x^2)^4 + \ldots$$

Now expand $(2x - x^2)^2$, $(2x - x^2)^3, \ldots$ and collect the terms: thus we shall obtain

$$(1+2x-x^2)^{\frac{1}{2}} = 1 + x - x^2 + x^3 - \frac{3}{2} x^4 + \ldots$$

Examples. XLII.

1. Write down the first three and the last three terms of $(a-x)^{13}$.

2. Write down the expansion of $(3-2x^2)^5$.

3. Expand $(1-2y)^7$.

4. Write down the first four terms in the expansion of $(x+2y)^n$.

5. Expand $(1+x-x^2)^4$.

6. Expand $(1+x+x^2)^5$.

7. Expand $(1-2x+x^2)^4$.

8. Find the coefficient of x^5 in the expansion of $(1+2x+3x^2)^7$.

9. Find the coefficient of x^6 in the expansion of $(1-2x+3x^2)^5$.

10. If the second term in the expansion of $(x+y)^n$ be 240, the third term 720, and the fourth term 1080, find x, y, and n.

11. If the sixth, seventh, and eighth terms in the expansion of $(x+y)^n$ be respectively 112, 7, and $\frac{1}{4}$, find x, y, and n.

12. Write down the first five terms of the expansion of $(a-2x)^{\frac{1}{4}}$.

13. Expand to four terms $\left(1-\frac{5}{6}x\right)^{-\frac{3}{5}}$.

14. Expand $(1-2x)^{-1}$.

15. Write down the coefficient of x^r in the expansion of $(1-x)^{-2}$.

16. Write down the sixth term in the expansion of $(3x-y)^{-\frac{3}{4}}$.

17. Expand to five terms $(a-3b)^{-\frac{10}{3}}$: shew that if $a=1$ and $b=\frac{1}{5}$ the fourth term is greater than either the third or the fifth.

18. Write down the coefficient of x^r in the expansion of $(1-x)^{-4}$.

19. Expand $(1+x+x^2)^{\frac{1}{2}}$ to four terms in powers of x.

20. Expand $(1-x+x^2)^{-\frac{1}{2}}$ to four terms in powers of x.

XLIII. *Scales of Notation.*

432. The student will of course have learned from Arithmetic that in the ordinary method of expressing whole numbers by figures, the number represented by each figure is always *some multiple of some power of ten.* Thus in 523 the 5 represents 5 hundreds, that is 5 times 10^2; the 2 represents 2 tens, that is 2 times 10^1; and the 3, which represents 3 units, may be said to represent 3 times 10^0; see Art. 324.

This mode of expressing whole numbers is called the *common scale of notation*, and *ten* is said to be the *base* or *radix* of the common scale.

433. We shall now shew that any positive integer greater than unity may be used instead of 10 for the radix; and then explain how a given whole number may be expressed in any proposed scale.

The figures by means of which a number is expressed are called *digits.* When we speak in future of *any radix* we shall always mean that this radix is some positive integer greater than unity.

434. *To shew that any whole number may be expressed in terms of any radix.*

Let N denote the whole number, r the radix. Suppose that r^n is the highest power of r which is not greater than N; divide N by r^n; let the quotient be a, and the remainder P: thus

$$N = ar^n + P.$$

Here, by supposition, a is less than r, and P is less than r^n. Divide P by r^{n-1}; let the quotient be b, and the remainder Q: thus

$$P = br^{n-1} + Q.$$

Proceed in this way until the remainder is less than r: thus we find N expressed in the manner shewn by the following identity,

$$N = ar^n + br^{n-1} + cr^{n-2} + \dots\dots + hr + k.$$

Each of the digits $a, b, c, \ldots\ldots h, k$ is less than r; and any one or more of them after the first may happen to be zero.

435. *To express a given whole number in any proposed scale.*

By a *given* whole number we mean a whole number expressed in words, or else expressed by digits in some assigned scale. If no scale is mentioned the common scale is to be understood.

Let N be the given whole number, r the radix of the scale in which it is to be expressed. Suppose $k, h, \ldots c, b, a$ the required digits, $n+1$ in number, beginning with that on the right hand: then

$$N = ar^n + br^{n-1} + cr^{n-2} + \ldots + hr + k.$$

Divide N by r, and let M be the quotient; then it is obvious that $M = ar^{n-1} + br^{n-2} + \ldots\ldots + h$, and that the remainder is k. Hence the first digit is found by this rule: *divide the given number by the proposed radix, and the remainder is the first of the required digits.*

Again, divide M by r; then it is obvious that the *remainder* is h; and thus the second of the required digits is found.

By proceeding in this way we shall find in succession all the required digits.

436. We shall now solve some examples.

Transform 32884 into the scale of which the radix is seven.

$$\begin{array}{r|l}
7 & 32884 \\
7 & 4697 \ldots 5 \\
7 & 671 \ldots 0 \\
7 & 95 \ldots 6 \\
7 & 13 \ldots 4 \\
 & 1 \ldots 6
\end{array}$$

Thus $32884 = 1 \cdot 7^5 + 6 \cdot 7^4 + 4 \cdot 7^3 + 6 \cdot 7^2 + 0 \cdot 7^1 + 5$, so that the number expressed in the scale of which the radix is seven is 164605.

Transform 74194 into the scale of which the radix is twelve.

$$
\begin{array}{r|l}
12 & 74194 \\
12 & 6182 \ldots 10 \\
12 & 515 \ldots 2 \\
12 & 42 \ldots 11 \\
 & 3 \ldots 6
\end{array}
$$

Thus $74194 = 3 \cdot 12^4 + 6 \cdot 12^3 + 11 \cdot 12^2 + 2 \cdot 12 + 10$.

In order to express the number in the scale of which the radix is twelve in the usual manner, we require two new symbols, one for *ten*, and the other for *eleven:* we will use *t* for the former, and *e* for the latter. Thus the number expressed in the scale of which the radix is twelve is 36*e*2*t*.

Transform 645032, which is expressed in the scale of which the radix is nine, into the scale of which the radix is eight.

$$
\begin{array}{r|l}
8 & 645032 \\
 & 72782 \ldots 4.
\end{array}
$$

The division by eight is performed thus: First eight is not contained in 6, so we have to find how often eight is contained in 64; here 6 stands for six times nine, that is fifty-four, so that the question is how often is eight contained in fifty-eight, and the answer is seven times with two over. Next we have to find how often eight is contained in 25, that is how often eight is contained in twenty-three, and the answer is twice with seven over. Next we have to find how often eight is contained in 70, that is how often eight is contained in sixty-three, and the answer is seven times with seven over. Next we have to find how often eight is contained in 73, that is how often eight is contained in sixty-six, and the answer is eight times with two over. Next we have to find how often eight is contained in 22, that is how often eight is contained in twenty, and the answer is twice with four over. Thus 4 is the first of the required digits.

We will indicate the remainder of the process; the student should carefully work it for himself, and then com-

pare his result with that which is here obtained.

```
8 | 72782
   8 | 8210 ... 2
   8 | 1023 ... 3
     8 | 113 ... 6
       8 | 12 ... 5
            1 ... 3.
```

Thus the number $= 1.8^6 + 3.8^5 + 5.8^4 + 6.8^3 + 3.8^2 + 2.8 + 4$, so that, expressed in the scale of which the radix is eight, it is 1356324.

437. It is easy to form an unlimited number of self-verifying examples. Thus, take two numbers, expressed in the common scale, and obtain their sum, their difference, and their product, and transform these into any proposed scale; next transform the numbers into the proposed scale, and obtain their sum, their difference, and their product in this scale; the results should of course agree respectively with those already obtained.

EXAMPLES. XLIII.

1. Express 34042 in the scale whose radix is five.

2. Express 45792 in the scale whose radix is twelve.

3. Express 1866 in the scale whose radix is two.

4. Express 2745 in the scale whose radix is eleven.

5. Multiply *e4t* by *te*; these being in the scale with radix twelve; transform them to the common scale and multiply them together.

6. Find in what scale the number 4161 becomes 10101.

7. Find in what scale the number 5261 becomes 40205.

8. Express 17161 in the scale whose radix is twelve, and divide it by *te* in that scale.

9. Find the radix of the scale in which 13, 22, 33 are in geometrical progression.

10. Extract the square root of *eet*001, in the scale whose radix is twelve.

XLIV. *Interest.*

438. The subject of Interest is discussed in treatises on Arithmetic; but by the aid of Algebraical notation the rules can be presented in a form easy to understand and to remember.

439. *Interest* is money paid for the use of money. The money lent is called the *Principal.* The *Amount* at the end of a given time is the sum of the Principal and the Interest at the end of that time.

440. Interest is of two kinds, *simple* and *compound.* When interest is charged on the Principal alone it is called *simple* interest; but if the interest as soon as it becomes due is added to the principal, and interest charged on the whole, it is called *compound* interest.

441. The *rate* of interest is the money paid for the use of a certain sum for a certain time. In *practice* the sum is usually £100, and the time is one year; and when we say that the rate is £4. 5*s.* per cent. we mean that £4. 5*s.*, that is £$4\frac{1}{4}$, is paid for the use of £100 for one year. In *theory* it is convenient, as we shall see, to use a symbol to denote the interest of *one* pound for one year.

442. *To find the amount of a given sum in any given time at simple interest.*

Let P be the number of pounds in the principal, n the number of years, r the interest of one pound for one year, expressed as a fraction of a pound, M the number of pounds in the amount. Since r is the interest of one pound for one year, Pr is the interest of P pounds for one year, and nPr is the interest of P pounds for n years; therefore

$$M = P + Pnr = P(1 + nr).$$

443. From the equation $M = P(1 + nr)$, if any three of the four quantities M, P, n, r are given, the fourth can be found: thus

$$P = \frac{M}{1 + nr}, \qquad n = \frac{M - P}{Pr}, \qquad r = \frac{M - P}{Pn}.$$

444. *To find the amount of a given sum in any given time at compound interest.*

Let P be the number of pounds in the principal, n the number of years, r the interest of one pound for one year, expressed as a fraction of a pound, M the number of pounds in the amount. Let R denote the amount of one pound in one year; so that $R=1+r$. Then PR is the amount of P pounds in one year. The amount of PR pounds in one year is PRR, or PR^2; which is therefore the amount of P pounds in *two* years. Similarly the amount of PR^2 pounds in one year is PR^2R, or PR^3, which is therefore the amount of P pounds in *three* years.

Proceeding in this way we find that the amount of P pounds in n years is PR^n; that is

$$M=PR^n.$$

The interest gained in n years is

$$PR^n-P \text{ or } P(R^n-1).$$

445. The *Present value* of an amount due at the end of a given time is that sum which with its interest for the given time will be equal to the amount. That is, the *Principal* is the present value of the *Amount;* see Art. 439.

446. *Discount* is an allowance made for the payment of a sum of money before it is due.

From the definition of *present value* it follows that a debt is fairly discharged by paying the *present value* at once: hence the discount is equal to the amount due diminished by its present value.

447. *To find the present value of a sum of money due at the end of a given time, and the discount.*

Let P be the number of pounds in the present value, n the number of years, r the interest of one pound for one year expressed as a fraction of a pound, M the number of pounds in the sum due, D the discount.

Let $R=1+r$.

At simple interest

$$M = P(1+nr), \text{ by Art. 442;}$$

therefore $$P = \frac{M}{1+nr}; \qquad D = M - P = \frac{Mnr}{1+nr}.$$

At compound interest

$$M = PR^n, \text{ by Art. 444;}$$

therefore $$P = \frac{M}{R^n}; \qquad D = M - P = \frac{M(R^n - 1)}{R^n}.$$

448. In practice it is very common to allow the *interest* of a sum of money paid before it is due instead of the *discount* as here defined. Thus at simple interest instead of $\frac{Mnr}{1+nr}$ the payer would be allowed Mnr for immediate payment.

Examples. XLIV.

1. At what rate per cent. will £a produce the same interest in one year as £b produces when the rate is £c per cent.?

2. Shew that a sum of money at compound interest becomes greater at a given rate per cent. for a given number of years than it does at twice that rate per cent. for half that number of years.

3. Find in how many years a sum of money will double itself at a given rate of simple interest.

4. Shew, by taking the first three terms of the Binomial series for $(1+r)^n$, that at five per cent. compound interest a sum of money will be more than doubled in fifteen years.

MISCELLANEOUS EXAMPLES.

1. Find the values when $a=5$ and $b=4$ of

$$a^3+3a^2b+3ab^2+b^3, \text{ of } a^2+10ab+9b^2, \text{ of } (a-b)^3,$$

and of $(a+9b)(a-b)$.

2. Simplify $5x-3[2x+9y-2\{3x-4(y-x)\}]$.

3. Square $3-5x+2x^2$.

4. Divide 1 by $1-x+x^2$ to four terms: also divide $1-x$ by $1-x^3$ to four terms.

5. Simplify $\dfrac{4x^3-17x+12}{6x^2-17x+12}$.

6. Find the L.C.M. of $4x^2-9$, $6x^2-5x-6$, and $6x^2+5x-6$.

7. Simplify $\dfrac{\frac{x}{a}+\frac{a}{x}-2}{x-a}+\dfrac{\frac{x}{a}+\frac{a}{x}+2}{x+a}$.

8. Solve $\dfrac{x-2}{3}+\dfrac{x+5}{6}=\dfrac{7x-6}{9}$.

9. The first edition of a book had 600 pages and was divided into two parts. In the second edition one quarter of the second part was omitted, and 30 pages were added to the first part; this change made the two parts of the same length. Find the number of pages in each part in the first edition.

10. In paying two bills, one of which exceeded the other by one third of the less, the change out of a £5 note was half the difference of the bills: find the amount of each bill.

11. Add together $y+\frac{1}{2}z-\frac{1}{3}x$, $z+\frac{1}{2}x-\frac{1}{3}y$, $x+\frac{1}{2}y-\frac{1}{3}z$; and from the result subtract $\frac{1}{6}x-y-\frac{1}{3}z$.

12. If $a=1$, $b=3$, and $c=5$, find the value of

$$\frac{2a^3+b^3+c^3+a^2(b-c)+b^2(2a-c)+c^2(2a+b)}{2a^3-b^3+c^3+a^2(b-c)-b^2(2a-c)+c^2(2a+b)}.$$

13. Simplify $(a+b)^2-(a+b)(a-b)-\{a(2b-2)-(b^2-2a)\}$.

14. Divide

$$2x^5-x^4y-4x^3y^2+5x^2y^3-4y^5 \text{ by } x^3-xy^2+2y^3.$$

15. Reduce to its lowest terms $\dfrac{x^4-2x^3+x^2-1}{x^4+x^2+1}$.

16. Find the L.C.M. of $x^2-9x-10$, $x^2-7x-30$, $(x+1)(x+3)(x-10)$, and x^2+4x+3.

17. Simplify

$$\frac{2}{x^2-9x-10}+\frac{3}{x^2-7x-30}-\frac{5}{x^2+4x+3}.$$

18. Solve $\qquad x-\dfrac{x-2}{3}=\dfrac{x+15}{4}-\dfrac{x}{5}$.

19. Solve $\dfrac{3}{2}(x-1)-\dfrac{2}{3}(x+2)+\dfrac{1}{4}(x-3)=4$.

20. Two persons A and B own together 175 shares in a railway company. They agree to divide, and A takes 85 shares, while B takes 90 shares and pays £100 to A. Find the value of a share.

21. Add together $a+2x-y+24b$, $3a-4x-2y-81b$, $x+y-2a+55b$;

and subtract the result from $3a+b+3x+2y$.

22. Find the value of $\dfrac{a^2b}{7}+\sqrt{7ab(2c^2-ab)}-(2a-3b)^3$, when $a=3$, $b=2\frac{1}{3}$, and $c=2$.

23. Simplify $\{x(x+a)-a(x-a)\}\{x(x-a)-a(a-x)\}$.

24. Divide $\dfrac{x^3}{6}-\dfrac{x}{4}+\dfrac{1}{8}-\dfrac{5x^2}{36}$ by $\dfrac{x}{3}-\dfrac{1}{2}$; and verify the result by multiplication.

25. Find the G.C.M. of x^4+3x^2-10 and x^4-3x^2+2.

26. Simplify $\frac{2a^2}{b^2-4a^2}-\frac{b}{b+2a}+\frac{a}{2a-b}$.

27. Find the L.C.M. of x^2-4, $4x^2-7x-2$, and $4x^2+7x-2$.

28. Solve $\frac{2x}{3}-\frac{x-1}{15}+\frac{\frac{1}{2}x-1}{6}=4$.

29. A man bought a suit of clothes for £4. 7*s*. 6*d*. The trowsers cost half as much again as the waistcoat, and the coat half as much again as the trowsers and waistcoat together. Find the price of each garment.

30. A farmer sells a certain number of bushels of wheat at 7*s*. 6*d*. per bushel, and 200 bushels of barley at 4*s*. 6*d*. per bushel, and receives altogether as much as if he had sold both wheat and barley at the rate of 5*s*. 6*d*. per bushel. How much wheat did he sell?

31. If $a=1$, $b=2$, $c=-\frac{1}{2}$, $d=0$, find the value of

$$\frac{a-b+c}{a-b-c}-\frac{ad-bc}{bd+ac}-\sqrt{\left(\frac{b^3}{a^3}-\frac{a^3}{c^3}\right)}.$$

32. Multiply together $x-a$, $x-b$, $x+a$, and $x+b$; and divide the result by $x^2+x(a+b)+ab$.

33. Divide $8x^5-x^2y^3+\frac{1}{2}y^5$ by $2x+y$.

34. Find the G.C.M. of $4x(x^2+10)-25x-62$ and $x^2-7x+10$.

35. Reduce to its lowest terms $\frac{12x^2-15xy+3y^2}{6x^3-6x^2y+2xy^2-2y^3}$.

36. Simplify $\frac{1}{1+\frac{a}{b+\frac{c}{d}}}+\frac{a}{a+b+\frac{c}{d}}$.

37. Solve $\frac{x-1}{9}-\frac{2-x}{4}-\frac{2x-1}{14}+\frac{2-3x}{30}=0$.

38. Solve $\frac{2x-1}{3}-\frac{x+4}{9}=\frac{5x-1}{27}$.

39. *A* can do a piece of work in one hour, *B* and *C* each in two hours: how long would *A*, *B*, and *C* take, working together?

40. *A* having three times as much money as *B* gave two pounds to *B*, and then he had twice as much as *B* had. How much had each at first?

41. Add together $2x+3y+4z$, $x-2y+5z$, and $7x-y+z$.

42. Find the sum, the difference, and the product of

$$3x^2-4xy+4y^2 \text{ and } 4x^2+2xy-3y^2.$$

43. Simplify

$$2a-3(b-c)+\{a-2(b-c)\}-2\{a-3(b-c)\}.$$

44. Find the G.C.M. of

$$x^4+67x^2+66 \text{ and } x^4+2x^3+2x^2+2x+1.$$

45. Simplify $\frac{x^4-1}{x^2-1}\times\frac{x+1}{x^4+2x^3+2x^2+2x+1}$.

46. Find the L.C.M. of x^2-4, x^2-5x+6, and x^2-9.

47. Reduce to its lowest terms $\frac{3x^3-4x^2-x-14}{6x^3-11x^2-10x+7}$.

48. Solve $3(x-1)-4(x-2)=2(3-x)$.

49. Solve $\sqrt{(9+4x)}=5-2\sqrt{x}$.

50. How much tea at 3*s*. 9*d*. per lb. must be mixed with 45 lbs. at 3*s*. 4*d*. per lb. that the mixture may be worth 3*s*. 6*d*. per lb.?

51. Multiply $3a^2+ab-b^2$ by $a^2-2ab-3b^2$, and divide the product by $a+b$.

52. Find the G.C.M. of $2x(x-3)+3(x-6\tfrac{2}{3})+15$ and $2x^3-5x^2-6x+15$.

53. Simplify $\frac{1}{1-\frac{1}{1+x}}+\frac{1}{1-\frac{1}{1-x}}$

54. Simplify $\frac{(a+b)^2}{a-b} \div \frac{ab+b^2}{a^2-ab}$.

55. Solve $\frac{1}{y}+\frac{2}{x}=\frac{2x+3}{xy}$, $\frac{1-2x^2}{x}=\frac{y}{x}-(1+2x)$.

56. Solve $x+\frac{3}{y}=\frac{7}{2}$, $3x-\frac{2}{y}=\frac{26}{3}$.

57. Solve $2(x-3)-\frac{1}{5}(y-3)=3$,

$$3(y-5)+\frac{1}{3}(x-2)=10.$$

58. Solve

$$7yz=10(y+z), \quad 3zx=4(z+x), \quad 9xy=20(x+y).$$

59. Solve $\frac{a}{x}+\frac{b}{y}=m$, $\frac{b}{x}-\frac{a}{y}=n$.

60. The denominator of a certain fraction exceeds the numerator by 2; if the numerator be increased by 5 the fraction is increased by unity: find the fraction.

61. Divide $x^5-\frac{1}{x^5}$ by $x-\frac{1}{x}$.

62. Reduce to its lowest terms $\frac{33x^2-49x-10}{21x^3-14x^2-29x-10}$.

63. Simplify $\left(a-\frac{2a}{x+\frac{1}{x}}\right) \div \left(\frac{x}{2}+\frac{1}{2x}-1\right)$.

64. Solve $3(x-1)+2(x-2)=x-3$.

65. Solve $\frac{x-1}{3}=\frac{y+1}{4}$, $\frac{2x-3}{5}=\frac{13-2y}{7}$.

66. Solve $5x+2=3y$, $6xy-10x^2+\frac{y-2x}{a}=8$.

67. Solve $\frac{x+y}{7}-\frac{2y-x}{3}=3$, $\frac{3y+2x}{4}+\frac{9(x-1)}{8}=\frac{x}{2}$.

68. Solve $\sqrt{(x^2+40)}=x+4$.

69. Solve $\frac{x^2+3x+2}{x+1}-\frac{x^2-x-6}{x+2}=\frac{5x}{2}$.

70. A father's age is double that of his son; 10 years ago the father's age was three times that of his son: find the present age of each.

71. Find the value when $x=4$ of

$$\sqrt{(2x+1)}-\left(x+\frac{6}{\sqrt{x}}\right)-\left(3-\frac{x^2}{4-\sqrt[3]{2x}}\right).$$

72. Reduce $\frac{3x^3-16x^2+23x-6}{2x^3-11x^2+17x-6}$ to its lowest terms; and find its value when $x=3$.

73. Resolve into simple factors x^2-3x+2, $x^2-7x+10$, and x^2-6x+5.

74. Simplify $\frac{1}{x^2-3x+2}+\frac{3}{x^2-7x+10}-\frac{4}{x^2-6x+5}$.

75. Solve $\frac{1}{14}\left(3x+\frac{11}{3}\right)-\frac{1}{7}(4x-2\frac{2}{3})=\frac{1}{2}(5x-1)$.

76. Solve $9x^2-63x+68=0$.

77. A man and a boy being paid for certain days' work, the man received 27 shillings and the boy who had been absent 3 days out of the time received 12 shillings: had the man instead of the boy been absent those 3 days they would both have claimed an equal sum. Find the wages of each per day.

78. Extract the square root of $9x^4-6x^3+7x^2-2x+1$; and shew that the result is true when $x=10$.

79. If $a : b :: c : d$, shew that

$$a^2c+ac^2 : b^2d+bd^2 :: (a+c)^3 : (b+d)^3.$$

80. If a, b, c, d be in geometrical progression, shew that a^2+d^2 is greater than b^2+c^2.

81. If n is a whole positive number $7^{2n+1}+1$ is divisible by 8.

82. Find the least common multiple of x^2-4y^2, $x^3+6x^2y+12xy^2+8y^3$, and $x^3-6x^2y+12xy^2-8y^3$.

83. Solve $\frac{3}{x}+\frac{1}{y}=\frac{1}{2}$, $\frac{4}{x}-\frac{3}{y}=2\frac{5}{6}$.

84. Solve $x^2+2x+2\sqrt{(x^2+2x+1)}=47$.

85. The sum of a certain number consisting of two digits and of the number formed by reversing the digits is 121; and the product of the digits is 28: find the number.

86. Nine gallons are drawn from a cask full of wine, and it is then filled up with water; then nine gallons of the mixture are drawn, and the cask is again filled up with water. If the quantity of wine now in the cask be to the quantity of water in it as 16 is to 9, find how much the cask holds.

87. Extract the square root of

$$16x^6+25y^6-30xy^5-24x^4y^2+9x^2y^4+40x^3y^3.$$

88. In an arithmetical progression the first term is 81, and the fourteenth is 159. In a geometrical progression the second term is 81, and the sixth is 16. Find the harmonic mean between the fourth terms of the two progressions.

89. If $\sqrt{5}=2\cdot23606$, find the value to five places of decimals of $\frac{6}{\sqrt{5}-1}$.

90. If x be greater than 9, shew that $\sqrt{x}$ is greater than $\sqrt[3]{(x+18)}$.

91. Divide $(x-y)^3-2y(x-y)^2+y^2(x-y)$ by $(x-2y)^2$.

92. Find the G.C.M. and the L.C.M. of

$$24(x^3+x^2y+xy^2+y^3) \text{ and } 16(x^3-x^2y+xy^2-y^3).$$

93. Simplify

$$\frac{x}{x^3+x^2y+xy^2+y^3}+\frac{y}{x^3-x^2y+xy^2-y^3}+\frac{1}{x^2-y^2}-\frac{1}{x^2+y^2}.$$

94. Solve $\frac{6x+7}{13}+\frac{2x+5}{7}=3-\frac{8x+1}{9}$.

95. Solve

$$xy+20(x-y)=0, \quad yz+30(y-z)=0, \quad 3x-2z=0.$$

96. Solve $3x^2-2x+\sqrt{(3x^2-4x-6)}=18+2x$.

97. *A* rows at the rate of $8\frac{1}{3}$ miles an hour. He leaves Cambridge at the same time that *B* leaves Ely. *A* spends 12 minutes in Ely and is back in Cambridge 2 hours and 20 minutes after *B* gets there. *B* rows at the rate of $7\frac{1}{2}$ miles an hour; and there is no stream. Find the distance from Cambridge to Ely.

98. An apple woman finding that apples have this year become so much cheaper that she could sell 60 more than she used to do for five shillings, lowered her price and sold them one penny per dozen cheaper. Find the price per dozen.

99. Sum to 8 terms and to infinity $12+4+1\frac{1}{3}+\ldots$

100. Find three numbers in geometrical progression such that if 1, 3, and 9 be subtracted from them in order they will form an arithmetical progression whose sum is 15.

101. Multiply $x^{\frac{7}{2}}-x^3+x^{\frac{5}{2}}-x^2+x^{\frac{3}{2}}-x+x^{\frac{1}{2}}-1$ by $x^{\frac{1}{2}}+1$; and divide $1-x^{\frac{3}{4}}$ by $1-x^{\frac{1}{4}}$.

102. Find the L.C.M. of x^3-a^3, x^3+a^3, $x^4+a^2x^2+a^4$, $x^3-ax^2-a^2x+a^3$, and $x^3+ax^2-a^2x-a^3$.

103. Simplify $\dfrac{a^3-b^3}{a^2-b^2+\dfrac{2b^2}{1+\dfrac{a+b}{a-b}}}$.

104. Solve

$$\frac{x+5}{6}+\frac{1}{9}\left(\frac{x}{2}+\frac{2}{5}\right)-\frac{2}{3}(3+2x)=\frac{4x-14}{3}+\frac{x+10}{10}.$$

105. Solve $\frac{6}{x-1}+\frac{8}{x-5}=\frac{7}{x+1}+\frac{18}{x+5}$.

106. Solve
$$x^2+y^2+z^2=50,$$
$$yz+xy-zx=7,$$
$$xy-yz-zx=47.$$

107. A and B travel 120 miles together by rail. B intending to come back again takes a return ticket for which he pays half as much again as A; and they find that B travels cheaper than A by 4*s.* 2*d.* for every 100 miles. Find the price of A's ticket.

108. Find a third proportional to the harmonic mean between 3 and $\frac{3}{7}$, and the geometric mean between 2 and 18.

109. Extract the square root of
$$\frac{x}{y}\left(2+\frac{x}{y}\right)-\frac{y}{x}\left(2-\frac{y}{x}+\frac{x}{y}\right).$$

110. If $a : b :: b : c$, shew that $b^4=\dfrac{a^2-b^2+c^2}{a^{-2}-b^{-2}+c^{-2}}$.

111. Divide $x^{\frac{3n}{2}}-x^{-\frac{3n}{2}}$ by $x^{\frac{n}{2}}-x^{-\frac{n}{2}}$.

112. Reduce $\dfrac{x^3+3x^2-20}{x^4-x^2-12}$ to its lowest terms, and find its value when $x=2$.

113. Solve $\dfrac{x-3}{x-4}-\dfrac{13}{3}=\dfrac{x+2}{3(6-x)}$.

114. Find the values of m for which the equation $m^2x^2+(m^2+m)ax+a^2=0$ will have its roots equal to one another.

115. Solve $3xy+x^2=10, \quad 5xy-2x^2=2$.

116. Solve $\dfrac{1}{x}+\dfrac{1}{y}=5, \quad \dfrac{x}{y}+\dfrac{y}{x}=2\frac{1}{6}$.

117. Find the fraction such that if you quadruple the numerator and add 3 to the denominator the fraction is doubled; but if you add 2 to the numerator and quadruple the denominator the fraction is halved.

118. Simplify $\{-(x^3)^{\frac{1}{2}}\}^{-\frac{1}{3}} \times \{-(-x)^{-3}\}^{\frac{1}{2}}$.

119. The third term of an arithmetical progression is 18; and the seventh term is 30: find the sum of 17 terms.

120. If $\frac{a+b}{2}$, b, $\frac{b+c}{2}$ be in harmonical progression, shew that a, b, c are in geometrical progression.

121. Simplify $a - \cfrac{1}{b + \cfrac{1}{b + \cfrac{ab}{a-b}}}$

122. Extract the square root of

$$37x^2y^2 - 30x^3y + 9x^4 - 20xy^3 + 4y^4.$$

123. Resolve $3x^3 - 14x^2 - 24x$ into its simple factors.

124. Solve $\frac{x+5}{2x-1} - \frac{3(5x+1)}{5x+4} = \frac{4}{2x-1} - 2\frac{1}{2}$.

125. Solve $x^3 + \frac{1}{x^3} = \frac{65}{8}$.

126. Solve $x^2 - y^2 = 9$, $x + 4 = 3(y-1)$.

127. Solve $y + \sqrt{(x^2-1)} = 2$, $\sqrt{(x+1)} - \sqrt{(x-1)} = \sqrt{y}$.

128. If a, b, c, d are in Geometrical Progression,

$$a : b+d :: c^3 : c^2d + d^3.$$

129. The common difference in an arithmetical progression is equal to 2, and the number of terms is equal to the second term: find what the first term must be that the sum may be 35.

130. Sum to n terms the series whose m^{th} term is 2×3^m.

131. Simplify $\frac{1+\sqrt{(1-2x)}}{1-\sqrt{(1-2x)}} + \frac{x-\sqrt{(1-2x)}}{x}$.

132. Find the G.C.M. of $30x^4 + 16x^3 - 50x^2 - 24x$ and $24x^4 + 14x^3 - 48x^2 - 32x$.

133. Solve $x^2-x-12=0$.

134. Form a quadratic equation whose roots shall be 3 and -2.

135. Solve $x^4+\frac{1}{x^4}=a^4+\frac{1}{a^4}$.

136. Solve $\frac{x^2}{\sqrt{(x^2+5)}}=1+\frac{1}{\sqrt{(x^2+5)}}$.

137. Having given $\sqrt{3}=1{\cdot}73205$, find the value of $\frac{6}{\sqrt{3}-1}$ to five places of decimals.

138. Extract the square root of $61-28\sqrt{3}$.

139. Find the mean proportional between $\frac{x+y}{x-y}$ and $\frac{x^2-y^2}{x^2y^2}$.

140. If a, b, c be the first, second and last terms of an arithmetical progression, find the number of terms. Also find the sum of the terms.

141. If d, c, b, a are 2, 3, 4, 5, find the values of

$$\frac{a+b+c}{a-b+c},\quad \frac{ab-cd}{ac-bd},\ \text{and}\ \sqrt{\frac{a-1}{b-3}}.$$

142. In the product of $1+4x+7x^2+10x^3+15x^4$ by $1+5x+9x^2+13x^3+17x^4$, find the coefficient of x^4.

Divide $21x^5-2x^4-70x^3-23x^2+33x+27$ by $7x^2+4x-9$.

143. Simplify $\frac{a^4-b^4}{a^2+b^2+2ab}\div\frac{a-b}{a^2+ab}$,

and
$$\frac{\sqrt{x}}{\sqrt{x}-\sqrt{a}}-\frac{\sqrt{a}}{\sqrt{x}+\sqrt{a}}-\frac{x-a}{x+a}.$$

144. Solve the following equations:

(1) $\frac{60-x}{14}-\frac{3x-5}{7}=6-\frac{24-3x}{4}$.

(2) $\frac{x+4}{x+3}=\frac{5x+12}{\frac{43x}{9}+9}$.

(3) $\frac{3x+5y}{20}+\frac{5x-3y}{8}=3,\quad \frac{x+1}{y+2}=\frac{2}{3}$.

145. Solve the following equations:

(1) $\frac{20}{8-x}+\frac{21}{6-x}=11$.

(2) $\sqrt{\frac{x}{2}}+\sqrt{3x+1}=7$.

(3) $3x^2-4xy=7,\quad 3xy-4y^2=5$.

146. A bill of £20 is paid in sovereigns and crowns, and 32 pieces are used: find how many there were of each kind.

147. A herd cost £180, but on 2 oxen being stolen, the rest average £1 a head more than at first: find the number of oxen.

148. Find two numbers when their sum is 40, and the sum of their reciprocals is $\frac{5}{48}$.

149. Find a mean proportional to $2\frac{1}{2}$ and $5\frac{5}{8}$; and a third proportional to 100 and 130.

150. If 8 gold coins and 9 silver coins are worth as much as 6 gold coins and 19 silver ones, find the ratio of the value of a gold coin to that of a silver coin.

151. Remove the brackets from

$$(x-a)(x-b)(x-c)-[bc(x-a)-\{(a+b+c)x-a(b+c)\}x].$$

152. Multiply $a+2\sqrt[4]{(a^2b)}+2\sqrt{b}$ by $a-2\sqrt[4]{(a^2b)}+2\sqrt{b}$.

153. Find the G.C.M. of $x^4-16x^3+93x^2-234x+216$ and $4x^3-48x^2+186x-234$.

154. Solve the following equations:

(1) $\frac{13x-1}{4}-\frac{28-5x}{3}=17-\frac{3x+1}{8}$.

(2) $\frac{2x+3}{3x+9}=\frac{2x-8}{3x-13}$.

(3) $x-y=3, \quad 3\left(\frac{1}{y}+\frac{1}{x}\right)=11\left(\frac{1}{y}-\frac{1}{x}\right)$.

155. Solve the following equations:

(1) $\sqrt{(x+1)}+\sqrt{(2x)}=7$.

(2) $7x-20\sqrt{x}=3$.

(3) $7xy-5x^2=36, \quad 4xy-3y^2=105$.

156. A boy spends his money in oranges; if he had bought 5 more for his money they would have averaged an half-penny less, if 3 fewer an half-penny more: find how much he spent.

157. Potatoes are sold so as to gain 25 per cent. at 6 lbs. for 5*d.*: find the gain per cent. when they are sold at 5 lbs. for 6*d.*

158. A horse is sold for £24, and the number expressing the profit per cent. expresses also the cost price of the horse: find the cost.

159. Simplify $\sqrt{\{4a^2+\sqrt{(16a^2x^2+8ax^3+x^4)}\}}$.

160. If the sum of two fractions is unity, shew that the first together with the square of the second is equal to the second together with the square of the first.

161. Simplify the following expressions:

$$a-[b-\{a+(b-a)\}],$$

$$25a-19b-[3b-\{4a-(5b-6c)\}]-8a,$$

$$[\{(a^{-m})^{-n}\}^{-p}]\div[\{(a^{2m})^{-3p}\}^{2n}].$$

162. Find the G.C.M. of $18a^3-18a^2x+6ax^2-6x^3$, and $60a^2-75ax+15x^2$.

163. Find the L.C.M. of $18(x^2-y^2)$, $12(x-y)^2$, and $24(x^3+y^3)$.

164. Solve the following equations:

(1) $\frac{2x-4}{7}+\frac{3x-2}{5}=7.$

(2) $\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}.$

(3) $\frac{\frac{x}{2}+4}{\frac{x}{3}+1}=\frac{2}{1}.$

(4) $2(x-y)=3(x-4y)$, $\quad 14(x+y)=11(x+8)$.

165. Solve the following equations:

(1) $32x-5x^2=12.$

(2) $\sqrt{(2x+3)}\sqrt{(x-2)}=15.$

(3) $x^2+y^2=290$, $\quad xy=143.$

(4) $3x^2-4y^2=8$, $\quad 5x^2-6xy=32.$

166. *A* and *B* together complete a work in 3 days which would have occupied *A* alone 4 days: how long would it employ *B* alone?

167. Find two numbers whose product is $\frac{2}{5}$ of the sum of their squares, and the difference of their squares is 96 times the quotient of the less number divided by the greater.

168. Find a fraction which becomes $\frac{1}{3}$ on increasing its numerator by 1, and $\frac{1}{4}$ on similarly increasing its denominator.

169. If $a : b :: c : d$, show that

$$\frac{1}{a}+\frac{1}{b} : \frac{1}{a}-\frac{1}{b} :: \frac{1}{c}+\frac{1}{d} : \frac{1}{c}-\frac{1}{d}.$$

170. Find a mean proportional between 169 and 256, and a third proportional to 25 and 100.

171. Remove the brackets from the expression

$$b-2\{b-3[a-4(a-b)]\}.$$

172. Simplify the following expressions:

$$\frac{x}{y}+\frac{2x^2+y^2}{xy}+\frac{3xy^2-3x^3-y^3}{x^2y}-\frac{4xy^3-2x^2y^2-y^4}{x^2y^2},$$

$$(p-q-m)p-(m+q-p)q+(q+m)m+m(p-m)+q^2,$$

$$\left(\frac{x^{p+q}}{x^q}\right)^p \div \left(\frac{x^q}{x^{q-p}}\right)^{p-q}.$$

173. Find the G.C.M. of $x^4+ax^3-9a^2x^2+11a^3x-4a^4$ and $x^4-ax^3-3a^2x^2+5a^3x-2a^4$.

174. Solve the following equations:

(1) $x-\frac{2x+1}{3}=\frac{x+7}{5}$.

(2) $\frac{10x+17}{18}-\frac{12x+2}{13x-16}=\frac{5x-4}{9}$.

(3) $9x+\frac{8y}{5}=70, \quad 7y-\frac{13x}{3}=44.$

(4) $\frac{6x+7}{3x+1}=\frac{2x+19}{x+7}$.

175. Solve the following equations:

(1) $x+4-\frac{7x-8}{x}=3.$

(2) $2x^2-3y^2=2, \quad xy=20.$

(3) $2y^2-x^2=1, \quad 3x^2-4xy=7.$

(4) $x+y=6, \quad x^3+y^3=126.$

176. When are the clock-hands at right angles first after 12 o'clock?

177. A number divided by the product of its digits gives as quotient 2, and the digits are inverted by adding 27: find the number.

178. A bill of £26. 15*s*. was paid with half-guineas and crowns, and the number of half-guineas exceeded the number of crowns by 17: find how many there were of each.

179. Sum to six terms and to infinity $12+8+5\frac{1}{3}+\ldots$.

180. Extract the square root of $55-7\sqrt{24}$.

181. If $x=\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$, and $y=\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$, find the value of x^2+xy+y^2.

182. Reduce to its lowest terms $\dfrac{3x^2-16x-12}{x^3-8x^2-12x+144}$.

183. If two numbers of two digits be expressed by the same digits in a reversed order, shew that the difference of the numbers can be divided by 9.

184. Solve the following equations:

(1) $\dfrac{3x-3}{4}-\dfrac{3x-4}{3}=\dfrac{21-4x}{9}$.

(2) $\dfrac{2x+3y}{6}+\dfrac{x}{3}=8, \quad \dfrac{7y-3x}{2}-y=11$.

(3) $4x-\dfrac{14-x}{x+1}=14$.

185 Solve the following equations:

(1) $\sqrt{(x+3)}\times\sqrt{(3x-3)}=24$.

(2) $\sqrt{(x+2)}+\sqrt{(3x+4)}=8$.

(3) $x^4-x^2(2x-3)=2x+8$.

186. Find two numbers in the proportion of 9 to 7 such that the square of their sum shall be equal to the cube of their difference.

187. A traveller sets out from A for B, going $3\frac{1}{2}$ miles an hour. Forty minutes afterwards another sets out from B for A, going $4\frac{1}{2}$ miles an hour, and he goes half a mile beyond the middle point between A and B before he meets the first traveller; find the distance between A and B.

188. Two persons A and B play at bowls. A bets B four shillings to three on every game, and after playing a certain number of games A is the winner of eight shillings. The next day A bets two to one, and wins one game more out of the same number, and finds that he has to receive three shillings. Find the number of games.

189. If $m = x - x^{-1}$ and $n = y - y^{-1}$,

shew that $mn + \sqrt{\{(m^2+4)(n^2+4)\}} = 2\left(xy + \frac{1}{xy}\right)$.

190. Sum to nineteen terms $\frac{9}{4} + \frac{3}{2} + \frac{3}{4} + \dots$.

191. Multiply $\frac{x^2}{2} - \frac{x}{3} + \frac{1}{4}$ by $\frac{x^2}{4} + \frac{x}{3} - \frac{1}{2}$.

Divide $\frac{3x^5}{4} - 4x^4 + \frac{77}{8}x^3 - \frac{43}{4}x^2 - \frac{33}{4}x + 27$ by $\frac{x^2}{2} - x + 3$.

192. Reduce to its lowest terms

$$\frac{4x^3 - 27x^2 + 58x - 39}{x^4 - 9x^3 + 29x^2 - 39x + 18}.$$

193. Find the L. C. M. of $x^3 + 2x^2y + 4xy^2 + 8y^3$ and $x^3 - 2x^2y + 4xy^2 - 8y^3$.

194. Solve the following equations:

(1) $\frac{1}{4}(x+6) - \frac{1}{12}(16 - 3x) = 4\frac{1}{6}$.

(2) $\frac{5x-9}{13} - \frac{23-2x}{9} = 3x - 20$.

(3) $\frac{1}{2}(x+y) = \frac{1}{3}(2x+4)$, $\frac{1}{3}(x-y) = \frac{1}{2}(x-24)$.

195. Solve the following equations:

(1) $\frac{3}{4}(x^2-3)=\frac{1}{8}(x-3)$.

(2) $\sqrt{(x+3)}+\sqrt{(3x-3)}=10$.

(3) $x+y=6,\quad (x^2+y^2)(x^3+y^3)=1440$.

196. The express train between London and Cambridge, which travels at the rate of 32 miles an hour, performs the journey in $2\frac{1}{4}$ hours less than the parliamentary train which travels at the rate of 14 miles an hour: find the distance.

197. Find the number, consisting of two digits, which is equal to three times the product of those digits, and is also such that if it be divided by the sum of the digits the quotient is 4.

198. The number of resident members of a certain college in the Michaelmas Term 1864, exceeded the number in 1863 by 9. If there had been accommodation in 1864 for 13 more students in college rooms, the number in college would have been 18 times the number in lodgings, and the number in lodgings would have been less by 27 than the total number of residents in 1863. Find the number of residents in 1864.

199. Extract the square root of

$$a^4-2a^3b+3a^2b^2-2ab^3+b^4,$$

and of $$(a+b)^4-2(a^2+b^2)(a+b)^2+2(a^4+b^4).$$

200. Find a geometrical progression of four terms such that the third term is greater by 2 than the sum of the first and second, and the fourth term is greater by 4 than the sum of the second and third.

201. Multiply $$8-3x+\frac{38x-6x^2-58}{7-2x}$$

by $$9-2x+\frac{7x^2-55+30x}{6-3x}.$$

202. Find the G.C.M. of x^4+4x^2+16 and x^4-x^3+8x-8.

203. Add together $\frac{1}{2+3x}$, $\frac{2x-5}{(2+3x)^2}$, $\frac{x^2-x+6}{(2+3x)^3}$.

Take $\frac{1}{1+x+x^2}$ from $\frac{1}{1-x+x^2}$.

204. Solve the following equations:

(1) $\frac{3x+5}{8}-\frac{21+x}{3}=39-5x$.

(2) $(a+b)(a-x)=a(b-x)$.

(3) $\frac{2x+3y}{16}+\frac{x}{12}=2\frac{3}{4}$, $\frac{7y-3x}{2}-2y=3$.

205. Solve the following equations:

(1) $6x+\frac{35-3x}{x}=44$.

(2) $4(x^2+3x)-2\sqrt{(x^2+3x)}=12$.

(3) $x^2+xy=15$, $y^2+xy=10$.

206. A person walked out from Cambridge to a village at the rate of 4 miles an hour, and on reaching the railway station had to wait ten minutes for the train which was then $4\frac{1}{2}$ miles off. On arriving at his rooms which were a mile from the Cambridge station he found that he had been out $3\frac{1}{4}$ hours. Find the distance of the village.

207. The tens digit of a number is less by 2 than the units digit, and if the digits are inverted the new number is to the former as 7 is to 4: find the number.

208. A sum of money consists of shillings and crowns, and is such, that the square of the number of crowns is equal to twice the number of shillings; also the sum is worth as many florins as there are pieces of money: find the sum.

209. Extract the square root of

$$4x^4+8ax^3+4a^2x^2+16b^2x^2+16ab^2x+16b^4.$$

210. Find the arithmetical progression of which the first term is 7, and the sum of twelve terms is 348.

211. Divide $6x^5-25x^4y+47x^3y^2-49x^2y^3+62xy^4-45y^5$ by $2x^2-7xy+9y^2$.

212. Multiply

$$3+5x-\frac{12+41x+36x^2}{4+7x} \text{ by } 5-2x+\frac{26x-8x^2-14}{3-4x}.$$

213. Reduce to its lowest terms

$$\frac{4x^3-45x^2+162x-185}{x^4-15x^3+81x^2-185x+150}.$$

214. Solve the following equations:

(1) $\frac{3x-2}{5}-\frac{1-5x}{11}=9.$

(2) $x+\frac{1}{4}y=17,\quad y+\frac{1}{4}x=8.$

(3) $\frac{1}{x}+\frac{1}{y}=\frac{1}{2},\quad \frac{1}{x}+\frac{1}{z}=\frac{4}{9},\quad \frac{1}{y}+\frac{1}{z}=\frac{5}{18}.$

215. Solve the following equations:

(1) $\frac{1}{x}-\frac{1}{x+3}=\frac{1}{6}.$

(2) $10xy-7x^2=7,\quad 5y^2-3xy=20.$

(3) $x+y=6,\quad x^4+y^4=272.$

216. Divide £34. 4*s*. into two parts such that the number of crowns in the one may be equal to the number of shillings in the other.

217. A number, consisting of three digits whose sum is 9, is equal to 42 times the sum of the middle and left-hand digits; also the right-hand digit is twice the sum of the other two: find the number.

218. A person bought a number of railway shares when they were at a certain price for £2625, and afterwards when the price of each share was doubled, sold them all but five for £4000: find how many shares he bought.

219. Four numbers are in arithmetical progression; their sum is 50, and the product of the second and third is 156: find the numbers.

220. Extract the square root of $17+12\sqrt{2}$.

221. Divide x^9-1 by x^3-1; and

$$m(qx^2-rx)+p(mx^3-nx^2)-n(qx-r) \text{ by } mx-n.$$

222. Simplify

$$\frac{ax^m-bx^{m+1}}{a^2bx-b^3x^3} \text{ and } \frac{a^2+b^2+c^2+2ab+2ac+2bc}{a^2-b^2-c^2-2bc}.$$

223. Find the L. C. M. of $7x^3-4x^2-21x+12$ and $21x^2-26x+8$.

224. Solve the following equations:

(1) $\frac{2x-4}{7}-\frac{2-3x}{5}=7.$

(2) $17x-13y=144, \quad 23x+19y=890.$

(3) $\frac{1}{x}-\frac{1}{y}=\frac{1}{8}, \quad \frac{1}{x}+\frac{1}{z}=\frac{1}{9}, \quad \frac{1}{z}-\frac{1}{y}=\frac{5}{72}.$

225. Solve the following equations:

(1) $\frac{x}{100}-\frac{21}{25x}=\frac{1}{4}.$

(2) $\cdot0075x^2+\cdot75x=150.$

(3) $\sqrt{(x+y)}+\sqrt{(x-y)}=\sqrt{c},$

$\quad b(x-a)+a(b-y)=0.$

226. A person walked out a certain distance at the rate of $3\frac{1}{2}$ miles an hour, and then ran part of the way back at the rate of 7 miles an hour, walking the remaining distance in 5 minutes. He was out 25 minutes: how far did he run?

227. A man leaves his property amounting to £7500 to be divided between his wife, his two sons, and his three daughters as follows: a son is to have twice as much as

a daughter, and the widow £500 more than all the five children together: find how much each person obtained.

228. A cistern can be filled by two pipes in $1\frac{1}{3}$ hours. The larger pipe by itself will fill the cistern sooner than the smaller by 2 hours. Find what time each will separately take to fill it.

229. The third term of an arithmetical progression is four times the first term; and the sixth term is 17: find the series.

230. Sum to n terms $3\frac{1}{3}+2\frac{1}{2}+1\frac{2}{3}+\ldots$

231. Simplify the following expressions:

$$x^{a+b+c}\times x^{a+b-c}\times x^{a-b+c}\times x^{b+c-a},$$

$$\frac{b}{a+b}-\frac{a+b}{2a}+\frac{a^2+b^2}{2a(a-b)},$$

$$\frac{a^2-ab+b^2}{a^3-3ab(a-b)-b^3}\times\frac{a^2-b^2}{a^2+b^2}.$$

232. Reduce to its lowest terms $\dfrac{x^2+11x+30}{9x^3+53x^2-9x-18}$.

233. Solve the following equations:

(1) $\dfrac{1}{x}+\dfrac{1}{2x}-\dfrac{1}{3x}=\dfrac{7}{3}$.

(2) $\dfrac{3}{1+x}+\dfrac{3}{1-x}=8$.

(3) $\dfrac{4x+5y}{40}=x-y,\quad \dfrac{2x-y}{3}+2y=\dfrac{1}{2}$.

234. Solve the following equations:

(1) $\dfrac{48}{x+3}=\dfrac{165}{x+10}-5$.

(2) $ax^2+b^2+c^2=a^2+2bc+2(b-c)x\sqrt{a}$.

(3) $\sqrt{(x+y)}+\sqrt{(x-y)}=4,\quad x^2+y^2=41$.

235. A body of troops retreating before the enemy, from which it is at a certain time 26 miles distant, marches 18 miles a day. The enemy pursues it at the rate of 23 miles a day, but is first a day later in starting, then after two days' march is forced to halt for one day to repair a bridge, and this they have to do again after two days' more marching. After how many days from the beginning of the retreat will the retreating force be overtaken?

236. A man has a sum of money amounting to £23. 15*s*. consisting only of half-crowns and florins; in all he has 200 pieces of money: how many has he of each sort?

237. Two numbers are in the ratio of 4 to 5; if one is increased, and the other diminished by 10, the ratio of the resulting numbers is inverted: find the numbers.

238. A colonel wished to form a solid square of his men. The first time he had 39 men over; the second time he increased the side of the square by one man, and then he found he wanted 50 men to complete it. Of how many men did the regiment consist?

239. Extract the square root of

$$a^6+2a^5b+3a^4b^2+4a^3b^3+3a^2b^4+2ab^5+b^6,$$

and of
$$a^2+4b^2+9c^2+4ab+6ac+12bc.$$

240. Multiply $x^{\frac{3}{2}}y^{\frac{1}{2}}-2xy+4x^{\frac{1}{2}}y^{\frac{3}{2}}$ by $x^{\frac{1}{2}}+2y^{\frac{1}{2}}$.

241. Simplify

$$40xy-(9x-8y)(5x+2y)-(4y-3x)(15x+4y),$$

and
$$\frac{1+x}{1-x}+\frac{1-x}{1+x}-\frac{1-x+x^2}{1+x^2}-\frac{1+x+x^2}{1-x^2}+2.$$

242. Find the G.C.M. of $x^4+ax^3+2a^2x^2+3a^3x+a^4$, and $x^4+ax^3+2a^2x^2+3a^3x+ab^2x+a^4+a^2b^2$.

243. Two shopkeepers went to the cheese fair with the same sum of money. The one spent all his money but 5*s*. in buying cheese, of which he bought 250 lbs. The other

bought at the same price 350 lbs., but was obliged to borrow 35*s.* to complete the payment. How much had they at first?

244. The two digits of a number are inverted; the number thus formed is subtracted from the first, and leaves a remainder equal to the sum of the digits; the difference of the digits is unity: find the number.

245. Find three numbers the third of which exceeds the first by 5, such that the product of their sum multiplied by the first is 48, and the product of their sum multiplied by the third is 128.

246. A person lends £1024 at a certain rate of interest; at the end of two years he receives back for his capital and compound interest on it the sum of £1156: find the rate of interest.

247. From a sum of money I take away £50 more than the half, then from the remainder £30 more than the fifth, then from the second remainder £20 more than the fourth part; at last only £10 remains: find the original sum.

248. Find such a fraction that when 2 is added to the numerator its value becomes $\frac{1}{3}$, and when 1 is taken from the denominator its value becomes $\frac{1}{4}$.

249. If I divide the smaller of two numbers by the greater, the quotient is ·21, and the remainder is ·04162; if I divide the greater number by the smaller the quotient is 4, and the remainder is ·742: find the numbers.

250. Shew that $\dfrac{(xy^2)^{\frac{1}{3}} - (x^2y)^{\frac{1}{3}} + x}{x+y} = \dfrac{x^{\frac{1}{3}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}}$.

251. Simplify

$$6a + [4a - \{8b - (2a + 4b) - 22b\} - 7b] \\ - [7b + \{8a - (3b + 4a) + 8b\} + 6a].$$

252. Multiply $a - x$ successively by $a + x$, $a^2 + x^2$, $a^4 + x^4$, $a^8 + x^8$; also multiply $a^{m-n}\, b^{n-p}$ by $a^{n-m}\, b^{p-n}\, c$.

253. Find the G.C.M. of $45a^3x+3a^2x^2-9ax^3+6x^4$ and $18a^2x-8x^3$.

254. Solve the following equations:

(1) $x-\frac{x-2}{3}=\frac{x+23}{4}-\frac{10+x}{5}$.

(2) $\frac{x}{6}+\frac{y}{11}=26, \quad \frac{x}{2}-\frac{y}{7}=46$.

(3) $a-x=\sqrt{\{a^2-x\sqrt{(4a^2-7x^2)}\}}$.

255. Divide the number 208 into two parts, such that the sum of one quarter of the greater and one third of the less when increased by 4, shall equal four times the difference of the two parts.

256. Two men purchase an estate for £9000. *A* could pay the whole if *B* gave him half his capital, while *B* could pay the whole if *A* gave him one-third of his capital: find how much money each of them had.

257. A piece of ground whose length exceeds the breadth by 6 yards, has an area of 91 square yards: find its dimensions.

258. A man buys a certain quantity of apples to divide among his children. To the eldest he gives half of the whole, all but 8 apples; to the second he gives half the remainder, all but 8 apples. In the same manner also does he treat the third and fourth child. To the fifth he gives the 20 apples which remain. Find how many he bought.

259. The sum of two numbers is 13, the difference of their squares is 39; find the numbers.

260. A horse-dealer buys a horse, and sells it again for £144, and gains just as many pounds per cent. as the horse had cost him. Find what he gave for the horse.

261. Simplify

$$(a+b)(a-b)-\{a+b-c-(b-a-c)+(b+c-a)\}(a-b-c).$$

262. Multiply $x^8+x^6+x^4+x^2+1$ by x^2-1; and $\frac{a}{x}-\frac{2x}{a}-1$ by $\frac{x}{a}-\frac{2a}{x}+1$.

263. What quantity, when multiplied by $x-\frac{1}{x}$, will give $x^3-\frac{1}{x^3}-\left(x-\frac{1}{x}\right)^2$?

264. Simplify the following expressions:

$$\frac{3x^3-13x^2+23x-21}{6x^3+x^2-44x+21},$$

$$\left\{\frac{a+b}{2(a-b)}-\frac{a-b}{2(a+b)}+\frac{2b^2}{a^2-b^2}\right\}\frac{a-b}{2b}.$$

265. Solve the following equations:

(1) $\frac{5x+3}{x-1}+\frac{2x-3}{2x-1}=6.$

(2) $\sqrt{(3+x)}+\sqrt{x}=\frac{6}{\sqrt{(3+x)}}.$

(3) $\frac{5x}{9}+9y=91,\quad \frac{5y}{9}+9x=167.$

266. Solve the following equations:

(1) $x^2-x-6=0.$

(2) $\frac{x+1}{x-1}+\frac{x+2}{x-2}=\frac{2x+13}{x+1}.$

(3) $x^2-xy+y^2=7,\quad x+y=5.$

267. The ratio of the sum to the difference of two numbers is that of 7 to 3. Shew that if half the less be added to the greater, and half the greater to the less, the ratio of the numbers so formed will be that of 4 to 3.

268. The price of barley per quarter is 15 shillings less than that of wheat, and the value of 50 quarters of barley exceeds that of 30 quarters of wheat by £7. 10*s.*: find the price per quarter of each.

269. Shew that

$$(bcd+cda+dab+abc)^2-(a+b+c+d)^2abcd$$
$$=(bc-ad)(ca-bd)(ab-cd).$$

270. Extract the square root of

$$x^4+x^3-\frac{5x^2}{12}-\frac{x}{3}+\frac{1}{9},$$

and of $$33-20\sqrt{2}.$$

271. If $a=y+z-2x$, $b=z+x-2y$, and $c=x+y-2z$, find the value of $b^2+c^2+2bc-a^2$.

272. Divide $x^4-21x+8$ by $1-3x+x^2$.

273. Add together $\frac{a+x}{a-x}$, $\frac{a-x}{a+x}$, and $\frac{a^2+x^2}{a^2-x^2}$.

Take $\frac{3a+x}{3a-x}$ from $\frac{27a^2+3ax+7x^2}{15a^2+ax-2x^2}$.

274. Multiply $3x-\frac{12ax-5x^2}{4a-3x}$ by $4x-\frac{20ax-7x^2}{5a-2x}$.

Divide $1-\frac{1}{1+x}$ by $1+\frac{x^2}{1-x^2}$.

275. Simplify $\dfrac{1}{a+\dfrac{1}{b+\dfrac{1}{c+d}}}$ and $\dfrac{\dfrac{1}{a^2}-\dfrac{1}{ax}+\dfrac{1}{x^2}}{\dfrac{1}{a^2}+\dfrac{1}{ax}+\dfrac{1}{x^2}}$.

276. Solve the following equations:

(1) $\frac{6}{x}-\frac{12}{x}+\frac{20}{x}=7.$

(2) $5y-3x=2,\quad 8y-5x=1.$

(3) $\frac{3x-2y}{4}-\frac{x-y}{2}=1,\quad \frac{x}{3}+\frac{y}{2}=4.$

277. Solve the following equations:

(1) $a^2(x-a)^2 = b^2(x+a)^2$.

(2) $\dfrac{x}{x-2} + \dfrac{5x+1}{x+3} = 5$.

(3) $\sqrt{(13x-1)} - \sqrt{(2x-1)} = 5$.

278. A person walked to the top of a mountain at the rate of $2\frac{1}{4}$ miles an hour, and down the same way at the rate of $3\frac{1}{2}$ miles an hour, and was out 5 hours: how far did he walk altogether?

279. Shew that the difference between the square of a number, consisting of two digits, and the square of the number formed by changing the places of the digits is divisible by 99.

280. If $a : b :: c : d$, shew that

$$\sqrt{(a^2+b^2)} : \sqrt{(c^2+d^2)} :: \sqrt[3]{(a^3+b^3)} : \sqrt[3]{(c^3+d^3)}.$$

281. Find the value of $\dfrac{\sqrt{\{a-(a-b)\}}}{\sqrt{(a^2+b^2)}} + \dfrac{\sqrt{\{5a-(a-b)\}}}{a+b}$, when $a = 3$, $b = 4$.

282. Subtract $(b-a)(c-d)$ from $(a-b)(c-d)$: what is the value of the result when $a = 2b$, and $d = 2c$?

283. Reduce to their simplest forms:

$$\frac{x^2-2ax-24a^2}{x^2-7ax-44a^2} \text{ and } \frac{x-y}{x+y} - \frac{x}{x-y} + \frac{y}{y-x}.$$

284. Solve the equations:

(1) $\dfrac{4}{3+x} - \dfrac{1}{x} = \dfrac{9}{7x}$.

(2) $\dfrac{3x-2y}{5} - \dfrac{x-y}{2} = 1$, $\dfrac{x}{3} + \dfrac{y}{2} = 4$.

(3) $\sqrt{(2x-1)} + \sqrt{(3x+10)} = \sqrt{(11x+9)}$.

285. Solve the equations:

(1) $10x + \frac{2}{1-x} = 9.$

(2) $\left(\frac{x}{a} - \frac{2a}{x} - 1\right)\left(1 + \frac{a}{x} - \frac{2x}{a}\right) = 0.$

(3) $x^2 - xy + y^2 = 7, \quad 5x - 2y = 9.$

286. In a time race one boat is rowed over the course at an average pace of 4 yards per second; another moves over the first half of the course at the rate of $3\frac{1}{2}$ yards per second, and over the last half at $4\frac{1}{2}$ yards per second, reaching the winning post 15 seconds later than the first. Find the time taken by each.

287. A rectangular picture is surrounded by a narrow frame, which measures altogether ten linear feet, and costs, at three shillings a foot, five times as many shillings as there are square feet in the area of the picture. Find the length and breadth of the picture.

288. If $a : b :: c : d$, show that

$$a+b+c+d : a+b-c-d :: a-b+c-d : a-b-c+d.$$

289. The volume of a pyramid varies jointly as the area of its base and its altitude. A pyramid, the base of which is 9 feet square, and the height of which is 10 feet is found to contain 10 cubic yards. Find the height of a pyramid on a base 3 feet square that it may contain 2 cubic yards.

290. Find the sum of n terms of the arithmetical progression $\frac{1}{1+x}, \ \frac{1}{1-x^2}, \ \frac{1}{1-x} \ldots$

291. Find the value of $a^3 - b^3 + c^3 + 3abc$, when $a = \cdot 03$, $b = \cdot 1$, $c = \cdot 07$.

292. Simplify $\frac{(ac-bd)^2+(ad+bc)^2}{c^2+d^2}-a^2$, and shew that

$$bc(b^2-c^2)+ca(c^2-a^2)+ab(a^2-b^2)$$
$$-(a+b+c)\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}=0.$$

293. If $a+b+c=0$, shew that $a^3+b^3+c^3=3abc$.

294. Reduce to its lowest terms

$$\frac{x^4+2x^3+6x-9}{x^4+4x^3+4x^2-9}.$$

295. Solve the following equations:

(1) $\frac{10x+17}{18}-\frac{12x+2}{13x-16}=\frac{5x-4}{9}$.

(2) $6x-5y=1, \quad y-x=12$.

(3) $\frac{x}{8}+8y=66, \quad \frac{y}{8}+8x=129$.

296. Solve the following equations:

(1) $\frac{x+1}{4}+\frac{3x+1}{x+4}=4$.

(2) $\sqrt{(2x+2)}\sqrt{(4x-3)}=20$.

(3) $\sqrt{(3x+1)}-\sqrt{(2x-1)}=1$.

297. A siphon would empty a cistern in 48 minutes, a cock would fill it in 36 minutes; when it is empty both begin to act: find how soon the cistern will be filled.

298. A waterman rows 30 miles and back in 12 hours, and he finds that he can row 5 miles with the stream in the same time as 3 against it. Find the times of rowing up and down.

299. Insert three Arithmetical means between $a-b$ and $a+b$.

300. Find x if $2^{x^2} : 2^{2x} :: 8 : 1$.

ANSWERS.

I. 1. 22. 2. 26. 3. 89. 4. 564.
5. 274. 6. 10. 7. 6. 8. 6. 9. 34.
10. 39. 11. 6. 12. 5. 13. 9. 14. 5.

II. 1. 55. 2. 81. 3. 94. 4. 8. 5. 27.
6. 81. 7. 12. 8. 11. 9. 21. 10. 15.
11. 10. 12. 3. 13. 2. 14. 127. 15. 6. 16. 1.

III. 1. 5. 2. 16. 3. 9. 4. 224. 5. 459.
6. 7. 7. 74. 8. 12. 9. 8. 10. 238.
11. 420. 12. 144. 13. 43. 14. 15. 15. 9. 16. 2.

IV. 1. 7. 2. 88. 3. 43. 4. 2. 5. 72.
6. 1. 7. 1. 8. 16. 9. 14. 10. 5. 11. 7.
12. 5. 13. 11. 14. 7. 15. 4. 16. 2.

V. 1. $15a-9b$. 2. $3x^2-3y^2$. 3. $9a+9b+9c$.
4. $4x+2y+4z$. 5. $a-b$. 6. $3x-3a-2b$. 7. $2a+2b$.
8. $a+b+c$. 9. $-2a+2b+2d$. 10. $2x^3-2x^2-8x+10$.
11. $5x^4+4x^3+3x^2+2x-9$. 12. $4a^3+2a^2b-4ab^2+b^3-7b^2$.
13. a^2x+3a^3. 14. $6ab-9a^2x+7ax^2+ax^3$. 15. $5x^2$.
16. $10x^2+8y^2+12x+12$. 17. x^4. 18. $x^3+y^3+z^3-3xyz$.

VI. 1. $3a+4b$. 2. $4a+2c$. 3. $a+5b+4c+d$.
4. $2x^2-2x-4$. 5. $3x^4-x^3-14x+18$.
6. $x^2-ax+2a^2$. 7. $-5xy-5xz+2y^2+yz$.
8. $3x^2+13xy-16xz-y^2-13yz$. 9. $2a^3-6a^2b+6ab^2-2b^3$.
10. $3x^3+4x+16$, x^3+8x^2.

VII. 1. a. 2. $2c$. 3. $a+a^3$. 4. $a-3b$.
5. $-2b+2c$. 6. $3x+3y-z$. 7. $a-b+c+d-e$.
8. $a-b+2c-d$. 9. $3c$. 10. $3a-3b$. 11. $2a-b$.
12. $5a$. 13. a. 14. $4a$. 15. $4a-16b-2c$.
16. $3a-2c$. 17. $9+3x$. 18. $7x+6$. 19. a.
20. $16-12x$. 21. $12x-15y$. 22. $4c$. 23. $3a-2c$.
24. $-8x^3-8x$.

VIII. 1. $8x^5$. 2. $12a^9$. 3. $4a^3b^3$.
4. $15x^7y^5z^3$. 5. $49x^4y^4z^4$. 6. $12a^3b-9ab^2$.
7. $24a^4-27a^3b$. 8. $6x^4y-8x^2y^3+10x^2yz^2$.
9. $x^4y^5z^2-x^2y^5z^6+x^4y^2z^6$.
10. $4x^2y^4z^4+6x^3y^5z^2-10x^4y^3z^3$. 11. $2x^2+3xy-2y^2$.
12. $6x^4-96$. 13. x^4-2x+1.
14. $1-2x-31x^2+72x^3-30x^4$. 15. $x^5-41x-120$.
16. $x^5+151x-234$. 17. $2x^5-18x^4+39x^3-25x^2+x+1$.
18. $x^6+1008x+720$.
19. $4x^6-5x^5+8x^4-10x^3-8x^2-5x-4$.
20. $x^8+2x^6+3x^4+2x^2+1$. 21. x^3-9a^2x.
22. $a^4+4a^3x+4a^2x^2-x^4$. 23. $-10b^3-ab^2+26a^2b-7a^3$.
24. $a^4-a^2b^2+2ab^3-b^4$. 25. $a^4+3a^2b^2+4b^4$.
26. $12x^3-17x^2y+3xy^2+2y^3$. 27. $x^6-x^4y^2+x^2y^4-y^6$.
28. $6x^4+17x^3y+26x^2y^2+19xy^3+4y^4$.
29. $x^3+y^3+3xy-2x-2y+1$. 30. x^5-32y^5.
31. $243x^5-y^5$. 32. $x^2-4y^2+12yz-9z^2$.
33. $a^3+a^2b+ab^2+b^3+2b^2x-(a-b)x^2$.
34. $a^3+b^3+c^3-3abc$. 35. $a^4+8b^2x^2(a^2-2)+16b^4x^4$.
36. $a^4-2a^2b^2+b^4+4abc^2-c^4$. 37. x^4-a^4.
38. $x^3+x^2(a+b+c)+x(ab+ac+bc)+abc$.
39. $x^8+x^4a^4+a^8$. 40. $x^4-5a^2x^2+4a^4$.

IX. 1. $5x^3$. 2. $-3a^3$. 3. $3xy$.
4. $-8a^2b^2c^2$. 5. $4a^4b^2y^2$. 6. x^2-2x+4.
7. $-a^2+4a-5$. 8. $x^2-3xy+4y^2$. 9. $5a^2b^2+ab-4$.
10. $15a^2b^2-12ab^3+9abc^2-5c^4$. 11. $x-4$. 12. $x-8$.
13. x^2+x+3. 14. $3x^2-2x+4$. 15. $3x^2+2x+1$.
16. x^2-3x+7. 17. $x^5+x^4+x^3+x^2+x+1$.
18. a^2+ab-b^2. 19. $x^3+3x^2y+9xy^2+27y^3$.
20. $x^3-x^2y+xy^2$. 21. $x^4+x^3y+x^2y^2+xy^3+y^4$.
22. $a^4-2a^3b+4a^2b^2-8ab^3+16b^4$.
23. $2a^3-6a^2b+18ab^2-27b^3$. 24. x^2+xy+y^2.
25. $x^2+2xy+3y^2$. 26. x^2-2x+2. 27. x^2-3x-1.
28. x^2-5x+6. 29. x^2-4x+8. 30. x^2+5x+6.

31. $x^3 - x - 19$. 32. $1 - 3x + 2x^2 - x^3$.
33. $x^4 + 2x^3 + 3x^2 + 2x + 1$. 34. $a^2 + 2ab + 3b^2$.
35. $a^3 + 2a^2b + 2ab^2 + b^3$. 36. $x^4 - 3x^2 + 4x + 1$.
37. $x^4 + 2x^3 + 3x^2 + 2x + 1$. 38. $x^8 - x^6 + 2x^2 - 2$.
39. $x - c$. 40. $ax^2 + bx + c$. 41. $x^2 - 2xy + y^2$.
42. $x^2 + x(y+1) + y^2 - y + 1$. 43. $7x + 4z$.
44. $a + b + c$. 45. $a + 2b + c$.
46. $a^2 + a(2b - c) + b^2 - bc + c^2$. 47. $a(b+c) - bc$.
48. $x^2 - x(a+b) + ab$. 49. $x + y - z$. 50. $x + y + z$.

X. 1. $225x^2 + 420xy + 196y^2$. 2. $49x^4 - 70x^2y^2 + 25y^4$.
3. $x^4 + 4x^3 - 8x + 4$. 4. $x^4 - 10x^3 + 39x^2 - 70x + 49$.
5. $4x^4 - 12x^3 - 7x^2 + 24x + 16$.
6. $x^2 + 4y^2 + 9z^2 + 4xy + 6xz + 12yz$. 7. $x^4 + 2x^3y + x^2y^2 - y^4$.
8. $x^4 + x^2y^2 + y^4$. 9. $x^4 - x^2y^2 - 2xy^3$
10. $x^4 - x^2y^2 + 2xy^3 - y^4$. 11. $x^6 + 2x^4 + 5x$
12. $x^4 - 18x^2 + 81$. 13. $a^4 -$ [illegible]$a^2b^2 - 4$
14. $16x^4 + 96x^3y + 144x^2y^2 - 81y^4$. 15. $a^4x^4 - b^4y$
16. $a^4x^4 - 2a^2b^2x^2y^2 + b^4y^4$.

XI. 1. $a^2 + b^2 + c^2$. 2. $a^2 + b^2 + c^2$.
3. $a^2 + b^2 + c^2 + d^2 + 2ac + 2bd$. 4. $6(a + b + c)$.
5. $2(a + b + c)$. 6. $2b(x + y)$. 7. $bx + ay + (a + b)z$.
8. $x(2a + c) + y(2b + a) + z(2c + b)$.
9. $2(a + b + c)(x + y + z)$.
10. $2(a^2 + b^2 + c^2 - ab - bc - ca)$. 11. $b - 11a$.
12. $b^2 - d^2$. 13. $2a + 4by$. 14. $(x + a)^2$. 15. a.
16. $2a - 5b + 4c$. 17. 6. 18. $x^3 + x^2y + xy^2 + y^3$.
19. $x^3 + x^2y + xy^2 + y^3$. 20. $12abc$. 21. $a + b + c + d$.
22. $3b$. 23. $9a^2 - 30ab + 25b^2$.
24. $-6c^2 + c(9a + 4b) - 6ab$. 25. $(x^2 + xy + y^2)^2$.
26. $(x^2 - xy + y^2)^2$. 27. $a^2 - 2ab + 3b^2$.
28. $x^2 - 8xy + 15y^2$. 29. $a^4 - a^2b^2 + b^4$. 30. $a^4 - b^4$.
31. $2a^2 - 3ab + 4b^2$. 32. $x - 1$. 33. $(x - 1)(x + 4)$.

34. $a+x$. 35. a^3+b^3. 36. x^2-ax+a^2.
37. $(x+4)(x+5)$. 38. $(x+5)(x+6)$.
39. $(x-5)(x-10)$. 40. $(x-10)^2$. 41. $(x-11)(x+12)$.
42. $(x+4)(x-11)$. 43. $(x-3)(x+3)(x^2+9)$.
44. $(x+5)(x^2-5x+25)$.
45. $(x-2)(x+2)(x^2+4)(x^4+16)$.
46. $(x-2)(x+2)(x^2+2x+4)(x^2-2x+4)$.
47. $(a+4b)(a+5b)$. 48. $(x-6y)(x-7y)$.
49. $(a+b-5c)(a+b-6c)$.
50. $(2x+2y-a-b)(x+y-3a-3b)$.

XII. 1. $3x^2$. 2. $4a^2b^2$. 3. $12x^4y^5z^4$.
4. $7a^2b^3x^3y^3$. 5. $2(x+1)$. 6. $3(x+1)$.
7. $4(a^2+b^2)$. 8. x^2-y^2. 9. $x+5$. 10. $x-7$.
11. $x-10$. 12. $x-12$. 13. x^2+3x+4.
14. x^2-5x+3. 15. x^2-6x+7. 16. x^2-6x-5.
17. $x+3$. 18. $x-4$. 19. x^2-x+1.
20. x^2-x+1. 21. $3x+2$. 22. x^2-x-1.
23. x^2-2. 24. $x-2$. 25. x^2+1.
26. x^2+3x+5. 27. $7x^2+8x+1$.
28. $x^4-2x^3+3x^2-2x+1$. 29. x^2-3x+1.
30. $x+1$. 31. $x+7$. 32. $x+3y$. 33. $x+a$.
34. $x-2a$. 35. $x-y$.

XIII. 1. $12a^2b^2$. 2. $36a^3b^2c^3$. 3. $24a^2b^2x^3y^3$.
4. $(a+b)(a-b)^2$. 5. $12ab(a^3+b^3)$. 6. $(a+b)(a^3-b^3)$.
7. $(x+1)(x+3)(x-4)$. 8. $(x+2)(x+4)(x^2+3x+1)$.
9. $x(2x+1)(3x-1)(4x+3)$.
10. $(x^2-5x+6)(x-1)(x-4)$.
11. $(x^2+3x+2)(x-3)(x+5)$.
12. $(x^2+x+1)(x^2+1)(x+1)(x-1)$.
13. $(x^3-x^2-4x+4)(x-1)(x-4)$.
14. $(x^2-ax+a^2)(x^2+ax+a^2)(x-a)^2$.
15. $36a^3b^3c^3$. 16. $120(a+b)^2(a-b)^2$.

17. $24(a-b)(a^3+b^3)$. 18. $105ab^2(a+b)(a-b)$.
19. x^6-1. 20. x^8-1. 21. $x^{12}-1$.
22. $(x+1)(x+2)(x+3)$. 23. $(x+1)(x+2)(x^2+2x-3)$.
24. $(x^3-19x-30)(x^2+5x+10)$.

XIV. 1. $3x+\frac{4x}{7}$. 2. $4ac+\frac{4c}{9}$. 3. $2a+\frac{3b}{4a}$.

4. $2x-\frac{5y}{6x}$. 5. $x+\frac{2}{x+3}$. 6. $2x-\frac{1}{x-3}$.

7. $x^2+3ax+3a^2+\frac{3a^3}{x-2a}$. 8. $x-1-\frac{2x-1}{x^2-x+1}$.

9. $x^3+x^2+x+1+\frac{2}{x-1}$. 10. x^3-x^2+x-1. 11. $\frac{4a^2}{3b}$.

12. $\frac{8(a^2+b^2)}{3(a+b)}$. 13. $\frac{3(a-b)}{2(a+b)}$. 14. $\frac{x^2}{(x-1)^2(x+1)}$.

15. $\frac{4x}{3y}$. 16. $\frac{3a+2b}{a+b}$. 17. $\frac{2(a-b)}{3(a+b)}$. 18. $\frac{(x^3-1)(x+1)}{x^2+1}$.

XV. 1. $\frac{2a^2x}{3y}$. 2. $\frac{a+b}{2b}$. 3. $\frac{a+b}{a-b}$. 4. $\frac{2ax}{ax-3y^2}$.

5. $\frac{4(a+b)}{5(a-b)}$. 6. $\frac{a^2-ab+b^2}{a-b}$. 7. $\frac{x+2}{x+5}$. 8. $\frac{x+7}{x-5}$. 9. $\frac{x+3}{x-7}$.

10 $\frac{x+b}{x+c}$. 11. $\frac{x-b}{x+c}$. 12. $\frac{3x-4}{4x-3}$. 13. $\frac{x+a-b-c}{x+b-a-c}$.

14. $\frac{x+3}{x^2-2x+5}$. 15. $\frac{x-3}{x^2+7x+3}$. 16. $\frac{x+5}{x^2+3x+2}$.

17. $\frac{x+7}{x^2-4x-3}$. 18. $\frac{6x-5}{3x^2+x+1}$. 19. $\frac{5x+4}{3x^2+x+2}$.

20. $\frac{x-a}{x^2-ax+a^2}$. 21. $\frac{x-4}{x+4}$. 22. $\frac{x^2+ax-2a^2}{2x^2+3ax+4a^2}$.

23. $\frac{x-3}{x^2-3x+1}$. 24. $\frac{x+a}{x^2+ax+a^2}$. 25. $\frac{x-3}{x^2+1}$.

26. $\frac{3x^2+x+2}{2x^2+x+3}$. 27. $\frac{3x(x^2-5a^2)}{2x^2+3a^2}$. 28. $\frac{x^2+1}{x^4+x^2+1}$.

29. $\frac{1}{x-1}$. 30. $\frac{x^3}{x^3-a^2y}$. 31. $\frac{1}{x^2-a^2}$. 32. $\frac{y^{n-1}}{x^{m+1}}$.

33. $\frac{9x^2}{12x^3},\dots$ 34. $\frac{4(x-1)}{4(x^2-1)},\dots$ 35. $\frac{a(x+a)}{x^2-a^2},\dots$

36. $\frac{a(a+b)(a^2+b^2)}{a^4-b^4},\dots$ 37. $\frac{(x-1)(x+1)^2}{(x^2-1)^2},\dots$

38. $\frac{a(x^2+ax+a^2)}{x^3-a^3},\dots$ 39. $\frac{x^2+ax+a^2}{x^4+a^2x^2+a^4},\dots$

40. $\frac{x-c}{(x-a)(x-b)(x-c)},\dots$

XVI. 1. $\frac{6a-6b-c}{4}$. 2. $\frac{2a}{a^2-b^2}$. 3. $\frac{a^2+2ab-b^2}{a^2-b^2}$.

4. $\frac{2cb}{a^2-b^2}$. 5. $\frac{a+b+c}{abc}$. 6. $\frac{1}{x-y}$. 7. $\frac{12x}{1-9x^2}$.

8. $\frac{a+x}{ax}$. 9. $\frac{a+b}{2a-2b}$. 10. $\frac{4a}{a+x}$. 11. $\frac{2a^2+9c^2}{6ac}$.

12. $\frac{b}{a-b}$. 13. $\frac{b(a+b)}{x^2-b^2}$. 14. $\frac{2x-3}{x(4x^2-1)}$. 15. $\frac{16}{(x-2)(x+2)^2}$.

16. $\frac{a}{a^2-b^2}$. 17. $\frac{a^4+6a^2x^2+x^4}{a^4-x^4}$. 18. $\frac{2}{(x+1)(x+2)(x+3)}$.

19. $\frac{5x^2-7x}{(x^2-1)(x-2)}$. 20. $\frac{4x^3}{y(x^2-y^2)}$. 21. $\frac{2x^2}{1-x^2}$. 22. $\frac{2x^2}{x^2-1}$.

23. $\frac{2a^2}{x(x^2-a^2)}$. 24. $\frac{2a^4+6a^2b^2}{a^4-b^4}$. 25. $\frac{3x^2}{x^2-1}$.

26. $\frac{4a^2(a^2-ax+x^2)}{a^4-x^4}$. 27. $\frac{4(x+10)}{x^4-16}$. 28. $\frac{2x^2-9x+44}{x^3+64}$.

29. $\frac{x^2-4ax-a^2}{(x^2-a^2)^2}$. 30. $\frac{2a}{x^2-a^2}$. 31. 1. 32. $\frac{x^2-2x}{x^3+1}$. 33. 0.

34. $\frac{6}{x(x+1)(x+2)}$. 35. $\frac{1}{(1+x^2)(1+x^3)}$. 36. $\frac{2x^2}{x^3+y^3}$.

37. $\frac{2y^2}{x^3-y^3}$. 38. $\frac{2x^3+2}{x^4+x^2+1}$. 39. $\frac{4(a^4x^3-b^4y^3)}{a^4x^4-b^4y^4}$.

40. $\frac{4x^3}{x^8+x^4+1}$. 41. 0. 42. $\frac{4a^3}{x^4-a^4}$. 43. $\frac{8b^7}{a^8-b^8}$.

44. $\frac{48a^3}{(x^2-a^2)(x^2-9a^2)}$. 45. $\frac{24b^4}{a(a^2-b^2)(a^2-4b^2)}$.

46. $\frac{c}{(x-a)(x-b)}$. 47. $\frac{x}{(x-a)(x-b)}$. 48. $\frac{x(a+b)-ab}{(x-a)(x-b)}$.

49. $\frac{1}{(a-c)(c-b)}$. 50. $\frac{c-a-b}{(c-a)(c-b)}$. 51. 0.

52. $-\frac{1}{c(c-a)(c-b)}$. 53. 1. 54. $\frac{3x-a-b-c}{(x-a)(x-b)(x-c)}$.

55. $\frac{3x^2-a^2-b^2-c^2}{(x-a)(x-b)(x-c)}$. 56. $\frac{1}{(x-a)(x-b)(x-c)}$.

XVII. 1. $\frac{4c}{5a}$. 2. 1. 3. $\frac{a^3b^3c^3}{x^3y^3z^3}$. 4. $\frac{1}{(x-1)(x+2)}$.

5. $x-a$. 6. $\frac{a^4-b^4}{ab}$. 7. $\frac{a^2b^2}{a^2-b^2}$. 8. $\frac{ax}{a^2-x^2}$.

9. $\frac{(x+y)^2}{x^2+y^2}$. 10. $\frac{x+c}{x+b}$. 11. $\frac{x}{x-y}$. 12. $\frac{(a-c)^2-b^2}{abc}$.

13. $\frac{x^6-ax^5+a^5x-a^6}{a^3x^3}$. 14. $\frac{x^2}{a^2}+\frac{a^2}{x^2}-\frac{y^2}{b^2}-\frac{b^2}{y^2}$. 15. 1.

XVIII. 1. $\frac{6ay}{bx}$. 2. $\frac{9c^2x^2}{16a^2z^2}$. 3. $\frac{1}{x+y}$. 4. $\frac{3(a-b)^2}{b(a+b)}$.

5. $\frac{x(a+2x)}{a^2}$. 6. $\frac{2x}{x-y}$. 7. $\frac{a+x}{x+y}$. 8. $\frac{x-b}{x-a}$.

9. $\frac{a+b-c}{c+a-b}$. 10. $\frac{1}{x^2-y^2}$. 11. $\left(\frac{x-1}{x-3}\right)^3$. 12. $\frac{y^4-x^4}{y^3}$.

13. $5x-1$. 14. $\frac{a^4+a^2+1}{a^2}$. 15. $\frac{(x^2+a^2)(x^4+a^4)}{x^3a^3}$.

16. $\frac{x^2-6a^2}{xa}$. 17. $\frac{x-y}{y}$. 18. $\frac{x^2+ax+a^2}{ax}$. 19. $\frac{a^2+x^2}{2ax}$.

20. $\frac{x^4-3x^3a+3a^3x+a^4}{a^2x^2}$. 21. 1. 22. $\frac{x-4}{x-5}$.

23. $\frac{1}{x+1}$. 24. $\frac{x^2-a^2}{x(a+b+c)-bc}$. 25. $\frac{1}{x+1}$.

26. $\frac{1+x}{1+x^2}$. 27. $x+1$. 28. $\frac{1+x^2}{1+x}$. 29. $\frac{(x^2+y^2)^2}{x^4+y^4}$.

30. x. 31. 1. 32. $\frac{(a^2+b^2)^2}{a^4+b^4}$. 33. $\frac{a^2}{b^2}$. 34. $\frac{b}{a}$.

35. 0. 36. $\frac{4}{9}$. 37. $2\frac{2}{7}$. 38. 0. 39. 0. 40. a.

XIX. 1. 6. 2. 9. 3. 7. 4. 11. 5. 21.
6. 2. 7. 4. 8. 7. 9. 8. 10. 5.
11. 18. 12. 6. 13. 2. 14. 27. 15. 15.
16. 63. 17. 60. 18. 36. 19. 64. 20. 96.
21. 45. 22. 24. 23. 120. 24. 72. 25. 12.
26. 6. 27. 5. 28. 1. 29. 6. 30. 2. 31. 2.
32. 3. 33. $1\frac{1}{2}$. 34. 7. 35. $1\frac{1}{5}$. 36. 11.
37. 5. 38. $2\frac{1}{3}$. 39. 3. 40. 7. 41. 11.
42. 12. 43. 4. 44. 3. 45. 7. 46. 3.
47. $5\frac{1}{2}$. 48. $1\frac{1}{3}$. 49. 10. 50. 6. 51. 10.
52. 7. 53. 1. 54. 12. 55. 5. 56. $\frac{1}{7}$.
57. 3. 58. 2. 59. 3. 60. 28. 61. 5.
62. 2. 63. 3. 64. 2. 65. 4. 66. 2.

XX. 1. 10. 2. 8. 3. 12. 4. 6.
5. -7. 6. 16. 7. 5. 8. $3\frac{1}{7}$. 9. -6.
10. 5. 11. 8. 12. $\frac{7}{4}$. 13. 3. 14. 2.
15. 7. 16. $1\frac{4}{7}$. 17. $\frac{1}{5}$. 18. 1. 19. 17.
20. 2. 21. 5. 22. 2. 23. 6. 24. 7. 25. 2.
26. 2. 27. 2. 28. $\frac{50}{29}$. 29. 7. 30. 4.
31. -1. 32. $\frac{3}{2}$. 33. -23. 34. 3. 35. $5\frac{1}{2}$.
36. $\frac{4}{13}$. 37. 0. 38. 20. 39. 3. 40. 5.
41. $a-b$. 42. $a+b$. 43. $b-a$. 44. $\frac{2ab}{a+b}$.
45. $2(a+b)$. 46. $\frac{a^2+ab+b^2}{a+b}$. 47. $\frac{ab}{a+b-c}$.
48. $\frac{ab(a+b-2c)}{(a+b)c-a^2-b^2}$. 49. $\frac{2ab}{a+b}$. 50. $\frac{a+b}{2}$.
51. $\frac{a+b+c+d}{m+n}$. 52. c. 53. $\frac{a^2}{b-a}$.

54. $\frac{ab-pq}{a+b+p+q}$. 55. $\frac{1}{2}(a+b+3)$. 56. $\frac{c^2-ab}{a+b-2c}$.

57. $\frac{2(a^2+ab+b^2)}{3(a+b)}$. 58. $\frac{1}{2}(a+b)$. 59. 4.

60. 50. 61. 25. 62. $\frac{13}{81}$. 63. $(a-b)^2$. 64. a.

XXI. 1. 30. 2. 2. 3. 13, 20. 4. 35, 50, 70. 5. 17, 31. 6. 28, 14. 7. 28. 8. November 20th. 9. 52. 10. 36, 27. 11. 48, 36. 12. 14, 24, 38. 13. 28, 32. 14. 103. 15. 54, 21. 16. 8. 17. 8, 12. 18. 10. 19. 36, 9. 20. 36, 12. 21. 100, 88. 22. 14. 23. 24, 76. 24. 21. 25. 36, 24. 26. 24, 60, 192. 27. 840. 28. 30000. 29. 420. 30. 24. 31. 500. 32. 10, 14, 18, 22, 26, 30. 33. 36, 26, 18, 12. 34. 50, 100, 150, 250. 35. 5, 6. 36. 24, 36, 56. 37. 88, 44. 38. 130, 150, 130, 90. 39. 13, 27. 40. 75, 25. 41. 85, 35. 42. 1000. 43. 18, 3, 3. 44. 24000. 45. 80. 46. 26, 16, 32, 27, 42. 47. £140. 48. $10\frac{1}{2}d$.

XXII. 1. 72. 2. 20, 30. 3. 200 miles from Edinburgh. 4. 12, 16. 5. 8, 16. 6. 32, 16. 7. 48. 8. 30. 9. 9, 16. 10. 30. 11. 18, 22, 10, 40. 12. 6, 24. 13. 10, 15, 3, 60. 14. 10 shillings. 15. 55, 45. 16. At the end of 56 hours. 17. 27, 17. 18. 168, 84, 42. 19. 16, 25, 7, 42. 20. 240, 180, 144 days. 21. 15, 21. 22. 2560. 23. 36, 54. 24. 60. 25. 12. 26. 8 pence. 27. 875, 1125. 28. 25. 29. 10, 20. 30. 20, 80. 31. $5\frac{5}{17}$. 32. 40, 50. 33. 11, 17. 34. 28. 35. 24. 36. 1024. 37. 450, 270. 38. 2200, 1620, 1100, 1080. 39. 60. 40. 7 + 12 + 32. 41. 30. 42. 60. 43. 240. 44. 3*d*. 9*d*. 1*s*. 4*d*. 45. 50*d*. 46. £$133\frac{1}{3}$. 47. 24. 48. 60. 49. £120000. 50. 25. 51. $4\frac{1}{2}$, $3\frac{1}{2}$. 52. 39. 53. 40.

54. 200000000. 55. 6*s*. 56. 48. 57. $49\frac{1}{11}$ minutes past three. 58. $32\frac{8}{11}$ minutes past three. 59. £288. 60. 2 seconds. 61. 40 minutes past eleven. 62. £300 and £200. 63. 14. 64. 640.

XXIII. 1. 10; 7. 2. 17; 19. 3. 2; 13. 4. 4; 1. 5. 5; 5. 6. 21; 12. 7. 20; 10. 8. 2; −3. 9. 3; 2. 10. 3; 2. 11. $3\frac{1}{2}$; 4. 12. 10; 7. 13. 19; 2. 14. $38\frac{1}{2}$; 70. 15. 6; 12. 16. $\frac{348}{157}$; $\frac{156}{157}$. 17. 10; 5. 18. 12; 12. 19. 20; 20. 20. 13; 5. 21. 9; 7. 22. 10; 4. 23. 4; 9. 24. 5; 7. 25. $2\frac{1}{2}$; 1. 26. ·2; ·2. 27. 10; 8. 28. 12; 3. 29. 3; 2. 30. 63; 14. 31. 3; 2. 32. 2; 3. 33. 4; 12. 34. a; b. 35. a; b. 36. $\frac{ab}{a+b}$; $\frac{ab}{a+b}$. 37. b; a. 38. $\frac{ab^2c}{a^2+b^2}$; $\frac{a^2bc}{a^2+b^2}$. 39. $\frac{ac}{a+b}$; $\frac{bc}{a+b}$. 40. $\frac{1}{a+b}$; 0. 41. a; b. 42. $a+b$; $a-b$. 43. $(a+b)^2$; $(a-b)^2$. 44. $\frac{c}{a+b}$; $\frac{c}{a+b}$.

XXIV. 1. 2; 1; 3. 2. 3; 4; 6. 3. 2; 1; 3. 4. 9; 11; 13. 5. 4; 0; 5. 6. 5; −5; 5. 7. 45; −21; 1. 8. 10; 7; 3. 9. 51; 76; 1. 10. $\frac{2}{3}$; $\frac{3}{4}$; $\frac{2}{5}$. 11. $x=\frac{1}{2}(b+c-a)$, &c. 12. $x=\frac{2}{3}(a+b+c)-a$, &c. 13. $x=\frac{1}{2}(b+c)$, &c. 14. $x=y=z=\frac{abc}{ab+bc+ca}$. 15. $x=a, y=b, z=c$. 16. $v=3, x=4, y=5, z=2$.

XXV. 1. 42; 26. 2. 12; 16. 3. 116; 166. 4. 24; 60. 5. 30*d.*; 8*d.* 6. 49; 21. 7. $\frac{4}{15}$. 8. 45; 63. 9. 72; 60. 10. 30*d.*; 15*d.* 11. 5*s.*; 3*s.*

12. 20; 52. 13. 70; 50. 14. $\frac{3}{5}$. 15. (24 – 1) 20. 16. 15; 65. 17. 12; 5. 18. 14; 10. 19. 24. 20. 1; 2. 21. 59. 22. 100 lbs. 23. 150 yards; 30, 20 yards per minute. 24. 21; 11. 25. 50; 75. 26. 70; 42; 35. 27. 90; 72; 60. 28. 12 miles. 29. 4 miles walking, 3 miles rowing, at first. 30. $33\frac{1}{3}$ miles per hour; $48\frac{1}{3}$ distance. 31. 45; 30 miles per hour. 32. 30; 50 miles per hour. 33. 60 miles; passenger train 30 miles per hour. 34. 150; 120; 90. 35. $3\frac{2}{3}$*s.*; 3*s.*; $2\frac{1}{3}$*s.* 36. 4; 59; 55. 37. 120; 80; 40. 38. 432. 39. 420; 35; 21 shillings. 40. 2; 4; 94.

XXVI. 1. ± 4. 2. ± 25. 3. ± 7. 4. ± 9. 5. ± 9. 6. ± 6. 7. 1, 2. 8. 2, 3. 9. 2, -12. 10. 3, $-\frac{1}{2}$. 11. $4\frac{1}{3}$, -3. 12. 10, 5. 13. 5, $-\frac{5}{2}$. 14. 6, -3. 15. $\frac{3}{2}$, $-\frac{1}{2}$. 16. $\frac{9}{2}$, $\frac{1}{2}$. 17. 5, $\frac{2}{3}$. 18. 3, -9. 19. $2\frac{1}{2}$, $-\frac{1}{2}$. 20. $1\frac{2}{3}$, $-1\frac{1}{2}$. 21. 1, 2. 22. 4. 23. 6, $\frac{9}{4}$. 24. 11, 3. 25. 5, $3\frac{1}{2}$. 26. 44, -2. 27. 7, $-\frac{7}{12}$. 28. 10, -10. 29. 3, $-2\frac{1}{3}$. 30. $\frac{1}{2}$, -3. 31. 2. 32. 2, -3. 33. ± 2. 34. 1, -4. 35. 3, $-\frac{2}{3}$. 36. 6, $2\frac{2}{5}$. 37. 6, $\frac{16}{7}$. 38. 7, $\frac{7}{3}$. 39. 8, $2\frac{4}{11}$. 40. 3, $-4\frac{2}{3}$. 41. 3, -5. 42. 3, $-\frac{5}{7}$. 43. 2, -1. 44. 4, -1. 45. 7, $3\frac{14}{15}$. 46. $1\frac{3}{4}$, 1. 47. $4\frac{1}{3}$, $\frac{1}{7}$. 48. 3, $-\frac{4}{5}$. 49. 3, -9. 50. -10, $9\frac{25}{29}$. 51. 3, $-1\frac{1}{3}$. 52. 3, $-1\frac{2}{3}$.

53. $4, 0.$ 54. $1\frac{1}{3}, 0.$ 55. $13, \frac{5}{7}.$ 56. $6, -3\frac{1}{3}.$
57. $5, -1\frac{5}{13}.$ 58. $5, 1\frac{1}{5}.$ 59. $5, -1\frac{1}{4}.$ 60. $2\frac{2}{3}, 0.$
61. $a \pm \frac{1}{a}.$ 62. $(a \pm b)^2.$ 63. $\pm \sqrt{(ab)}.$ 64. $a, -\frac{b(a+b)}{2a+b}.$

XXVII. 1. $\pm 2, \pm 3.$ 2. $49.$ 3. $4.$ 4. $\pm 4.$
5. $5, -3.$ 6. $3, -2.$ 7. $6, 0.$ 8. $12, -3.$
9. $9, -12.$ 10. $\pm 3.$ 11. $2, -15\frac{1}{3}.$ 12. $4, 11\frac{1}{8}.$
13. $1\frac{2}{5}.$ 14. $16.$ 15. $1.$ 16. $\frac{3}{5}, \frac{4}{5}.$ 17. $4.$
18. $4.$ 19. $\frac{4(a+b)(a^2+b^2)}{(a-b)^2}.$ 20. $\frac{a-1}{2}.$ 21. $3a^2.$
22. $0, \pm \frac{1}{\sqrt{5}}.$ 23. $0, \pm 5.$ 24. $0, \pm \sqrt{2}.$ 25. $2, \pm 1.$
26. $0, \pm \sqrt{(ab)}.$ 27. $a, -2a, -2a.$ 28. $a, \frac{3a}{2}, -\frac{a}{2}.$

XXVIII. 1. 36, 24. 2. 36, 24. 3. 30, 24.
4. 18, 12, 9. 5. 12, 10. 6. 4, 6. 7. 196.
8. 3, 48. 9. 11. 10. 7. 11. 6, 12. 12. 15.
13. 24. 14. 27 lbs. 15. 8*s.* 9*d.*, 7*s.* 16. £20.
17. 126, 96. 18. 8*d.* 19. 10, 9 miles. 20. 56.
21. 192, 128. 22. 9 gallons. 23. 64. 24. Equal.
25. 4 per cent.

XXIX. 1. $5, -4;\ 4, -5.$ 2. $4, -\frac{25}{7};\ 1, -\frac{71}{35}.$
3. $\pm 8;\ \pm 6.$ 4. $6, 12;\ 2, -4.$ 5. $7, -4;\ 4, -7.$
6. $4, -\frac{48}{13};\ 3, -\frac{41}{13}.$ 7. $-24, \frac{6}{5};\ 12, \frac{4}{5}.$ 8. $6, -\frac{4}{81};\ 5, \frac{13}{81}.$
9. $2, -\frac{29}{24};\ 4, -\frac{53}{6}.$ 10. $6, 0;\ 5, 0.$ 11. $\frac{2}{3}, 0;\ \frac{3}{2}, 0.$
12. $3, 6;\ \frac{1}{3}, \frac{2}{3}.$ 13. $4, \frac{1}{8};\ 8, \frac{1}{4}.$ 14. $\frac{a+b}{a}, 0;\ \frac{a+b}{b}, 0.$ 15. $a, b.$

16. $a, \frac{(3b-a)a}{a+b}$; $b, \frac{(3a-b)b}{a+b}$. 17. $a, \frac{2ab^2}{a^2+b^2}$; $b, \frac{2ba^2}{a^2+b^2}$.

18. $a, 0$; $0, b$. 19. $\pm 4, \pm \frac{7}{\sqrt{2}}$; $\pm 3, \frac{1}{\pm\sqrt{2}}$. 20. ± 5; ± 4.

21. ± 7; ± 6. 22. ± 15; ± 7. 23. $\pm 4, \pm 14$; $\pm 1, \mp 4$.

24. ± 9; ± 4. 25. $\pm 3, \pm 36$; $\pm 5, \mp \frac{23}{2}$. 26. ± 9; ± 3.

27. ± 8; ± 6. 28. ± 2; ± 1. 29. $\pm 9, \pm 8\sqrt{2}$; $\pm 7, \pm\sqrt{2}$.

30. ± 4; ± 1. 31. $0, 1, \frac{15}{22}$; $0, 2, \frac{9}{22}$.

32. $\pm\frac{(a+1)b}{\sqrt{(2a^2+2)}}$; $\pm\frac{(a-1)b}{\sqrt{(2a^2+2)}}$. 33. $\pm a, \pm\frac{a+b}{\sqrt{2}}$; $\pm b, \pm\frac{a-b}{\sqrt{2}}$.

34. $\pm a, \pm\frac{a+1}{\sqrt{2}}$; $\pm 1, \pm\frac{a-1}{\sqrt{2}}$. 35. $6, -4$; $4, -6$.

36. $5, 4$; $4, 5$. 37. $4, 2$; $2, 4$. 38. $4, -3$; $3, -4$.

39. $1, 2$; $2, 1$. 40. $\pm 4, \pm 3$; $\pm 3, \pm 4$. 41. $2, 1$; $\frac{2}{3}, \frac{1}{3}$.

42. ± 5; ± 3. 43. $2, 1, -1, -2$; $1, 2, -2, -1$.

44. $\frac{1}{2}, \frac{-2\pm\sqrt{3}}{2}, \frac{-1\pm\sqrt{13}}{4}$; $1, -2\mp\sqrt{3}, \frac{-1\mp\sqrt{13}}{2}$.

45. $3, -\frac{1}{3}$; $6, -\frac{2}{3}$. 46. $5, -\frac{5}{3}$; $2, -\frac{2}{3}$.

47. 2; 1. 48. $4, \frac{3}{2}$; $\frac{1}{4}, -\frac{9}{4}$; $2, \frac{9}{2}, -\frac{7}{4}, \frac{3}{4}$.

49. $a+b+1, -\frac{a+b+1}{a+1}$; $b, -\frac{b}{a+1}$. 50. $\pm\frac{a}{3}$; $\pm 3b$.

51. $\pm\frac{a}{4}$; $\pm 2b$. 52. $0, a+b, \frac{1}{2}(a-b)\pm\frac{1}{2}\sqrt{\{(a+3b)(a-b)\}}$; $0, a+b, \frac{1}{2}(a-b)\mp\frac{1}{2}\sqrt{\{(a+3b)(a-b)\}}$. 53. $x = a \div \sqrt[4]{(abc)}$; &c.

54. $(x+y)(y+z)(z+x) = \pm abc$; &c. 55. ± 1; ± 2; ± 3.

56. $\frac{8}{3}, \frac{3}{2}$; $\frac{3}{2}, \frac{8}{3}$; ± 2.

XXX. 1. 11; 7. 2. 6; 18. 3. 8; 24. 4. 8; 16. 5. 10; 15. 6. 10; 12. 7. 7; 5. 8. 18; 8: 6; 16. 9. 5; 3. 10. 4; 2. 11. 2; 2. 12. 4; 6. 13. 7; 4. 14. 12; 8. 15. 20; 15. 16. 30; 40. 17. 60; 10. 18. 6·4. 19. 160; £2. 20. 24; 4*s.*; 3*s.* 21. 756; 36; 27. 22. $4\frac{1}{2}$ walking; $4\frac{1}{2}$ rowing at first. 23. 10; 12 miles per hour. 24. 6 miles.

XXXI. 1. $8x^6y^9z^{12}$. 2. $-8x^6y^6z^9$. 3. $81a^4b^8c^{12}$.
4. $\frac{4x^4}{9y^4}$. 5. $-\frac{64x^3}{27y^6}$. 6. $\frac{x^{12}}{y^6z^3}$.
7. $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$.
9. $a^6-3a^4b^2+3a^2b^4-b^6$. 10. $1-3x+3x^2-x^3$.
11. $8+12x+6x^2+x^3$. 12. $27-54x+36x^2-8x^3$.
13. $1+4x+6x^2+4x^3+x^4$. 14. $x^4-8x^3+24x^2-32x+16$.
15. $16x^4+96x^3+216x^2+216x+81$. 16. $2a^3x^3+6axb^2y^2$.
17. $2a^4x^4+12a^2x^2b^2y^2+2b^4y^4$. 18. $2(5x+10x^3+x^5)$.
19. $1-4x^2+6x^4-4x^6+x^8$. 20. $1+2x+3x^2+2x^3+x^4$.
22. $1+2x-x^2-2x^3+x^4$. 23. $1+6x+13x^2+12x^3+4x^4$.
24. $1-6x+15x^2-18x^3+9x^4$. 25. $2(4+25x^2+16x^4)$.
26. $1+3x+6x^2+7x^3+6x^4+3x^5+x^6$.
28. $1+3x-5x^3+3x^5-x^6$.
29. $1+9x+33x^2+63x^3+66x^4+36x^5+8x^6$.
30. $1-9x+36x^2-81x^3+108x^4-81x^5+27x^6$.
31. $2(36x+171x^3+144x^5)$. 32. $1-2x+3x^2-x^4+2x^5+x^6$.
33. $1+4x+10x^2+20x^3+25x^4+24x^5+16x^6$.
34. $4(ab+ad+bc+cd)$. 35. $2(a^2+2ac+c^2+b^2+2bd+d^2)$.
36. $1+6x+15x^2+20x^3+15x^4+6x^5+x^6$.
37. $1-12x+60x^2-160x^3+240x^4-192x^5+64x^6$.
38. $1+8x+28x^2+56x^3+70x^4+56x^5+28x^6+8x^7+x^8$.
39. $1-3x^3+3x^6-x^9$. 40. $1+3x^2+6x^4+7x^6+6x^8+3x^{10}+x^{12}$.

XXXII. 1. $3a^2b^2$. 2. $2ab$. 3. $-4ab^2$. 4. $2ab^2c^3$.
5. $-ab^2c^3$. 6. $\frac{5ab}{7c^2}$. 7. $-\frac{6ab^3}{5c^2}$. 8. $\frac{3a}{bc}$. 9. $\frac{a}{2b^2}$.

10. $\frac{2ab^2}{c^4}$. 11. $4a+5b$. 12. $7a^2-6b$. 13. $6x^3+1$.
14. $8a+3bc$. 15. $\frac{5a+2b}{5a+2c}$. 16. $\frac{3x^2-4}{2x-3}$. 17. x^2+x+1.
18. $1-x+2x^2$. 19. x^2+3x+8. 20. x^2-2x-2.
21. $1-2x+3x^2$. 22. $2x^4-x^2-2$. 23. $x^2-ax+2a^2$.
24. x^2-ax+b^2. 25. $x^3-6x^2+12x-8$. 26. $x^3+2ax^2-2a^2x-a^3$.
27. $1-x+x^2-x^3+x^4$. 28. $\frac{2x}{3y}-\frac{4x}{5z}-\frac{3y}{4z}$. 29. $1+x$.
30. $2x-3y$. 31. $1-x+x^2$. 32. $x^2-(a+b)x+ab$.
33. $x+1$. 34. x^2-xy+y^2. 35. 34. 36. 45.
37. 61. 38. 72. 39. 87. 40. 99.
41. 123. 42. 321. 43. 407. 44. 55·5.
45. 6·42. 46. ·914. 47. 1234. 48. 5420.
49. 620·1. 50. 70·58. 51. 8·008. 52. ·4937.
53. 12007. 54. 504·06. 55. 1·8042. 56. 2·1319.
57. ·75416. 58. 443329. 59. ·94868. 60. 2·49198.
61. ·65574. 62. ·09233. 63. 4·12310. 64. 11·35781.
65. 18·63488. 66. 119·56331. 67. $2x+3y$.
68. $12x^2+4y^3$. 69. $x-a-b$. 70. x^2+x+1.
71. x^2-ax-a^2. 72. $2x^2+4cx-3c^2$. 73. $1-3x+4x^2$.
74. $1-x+x^2-x^3$. 75. $1+2x$. 76. $3x-1$.
77. 27. 78. 35. 79. 54. 80. 61. 81. 88. 82. 92.
83. 138. 84. 148. 85. 378. 86. 39·2. 87. 5·76.
88. ·604. 89. 1111. 90. 2755. 91. 45045. 92. 17479.

XXXIII. 1. $\frac{1}{3}$. 2. $\frac{1}{8}$. 3. $\frac{1}{10}$. 4. 100. 5. $\frac{1}{27}$.
6. a^{-6}. 7. a^6. 8. a^{-2}. 9. a^{-1}. 10. $a^{\frac{7}{12}}$. 11. $x^{\frac{3}{2}}-y^{\frac{3}{2}}$.
12. $a-b$. 13. $x^2+2x^{\frac{3}{2}}+x-4$. 14. x^4+1+x^{-4}. 15. $a^{-1}-1$.
16. $a^2-3a^{\frac{2}{3}}+3a^{-\frac{2}{3}}-a^{-2}$. 17. $a^2+2a^{\frac{3}{2}}b^{\frac{1}{2}}+ab-x^{\frac{2}{3}}y^{\frac{4}{3}}$.
18. $x^{\frac{5}{2}}+x^{\frac{3}{2}}y-xy^{\frac{3}{2}}-y^{\frac{5}{2}}$. 19. $x^{\frac{1}{2}}+x^{\frac{1}{3}}y^{\frac{1}{6}}+x^{\frac{1}{6}}y^{\frac{1}{3}}+y^{\frac{1}{2}}$.
20. $a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$. 21. $16x^{-\frac{2}{3}}-12x^{-\frac{1}{3}}y^{-\frac{2}{3}}+9y^{-\frac{4}{3}}$.

22. $x+y$. 23. $a^{\frac{1}{3}}-a^{\frac{1}{6}}b^{\frac{1}{6}}+b^{\frac{1}{3}}$. 24. $a^{\frac{1}{3}}+b^{\frac{1}{3}}-c^{\frac{1}{3}}$.
25. $x^{\frac{1}{2}}+2x^{\frac{3}{8}}a^{\frac{1}{2}}+3x^{\frac{1}{4}}a+2x^{\frac{1}{8}}a^{\frac{3}{2}}+a^2$. 26. $x^{\frac{1}{4}}-2x^{\frac{1}{8}}y^{\frac{1}{8}}+y^{\frac{1}{4}}$.
27. $x^{\frac{1}{4}}-2x^{-\frac{1}{4}}$. 28. $x-2-x^{-1}$. 29. $x^{\frac{5}{6}}-2x^{\frac{1}{2}}+x^{\frac{1}{3}}$.
30. $2x^{\frac{3}{4}}-3+4x^{-\frac{3}{4}}$.

XXXIV. 1. $7\sqrt{2}$. 2. $9\sqrt[3]{4}$. 3. $\frac{8}{3}\sqrt{3}$. 4. $\frac{\sqrt[3]{4}}{4}$.
5. $\frac{13\sqrt{15}}{10}$. 6. $\frac{5\sqrt[3]{2}}{2}$. 7. $2+2\sqrt{2}-2\sqrt{3}$. 8. $2+\frac{5}{6}\sqrt{6}$.
9. $4+\frac{5}{2}\sqrt{2}$. 10. $5+2\sqrt{6}$. 11. $\frac{24-\sqrt{15}}{33}$.
12. $\frac{1}{7}\left(18+9\sqrt{6}+4\sqrt{15}+6\sqrt{10}\right)$. 13. $3+\sqrt{5}$.
14. $3-\sqrt{7}$. 15. $\sqrt{6}+\sqrt{2}$. 16. $\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}$.
17. $\sqrt{3}-\sqrt{2}$. 18. $2+\sqrt{3}$. 19. $\sqrt{3}$. 20. $\sqrt{10}$.

XXXV. 1. $\frac{2}{9}$. 2. $\frac{7}{12}, \frac{5}{8}, \frac{2}{3}, \frac{3}{4}, \frac{8}{9}$. 3. $\frac{5}{27}$.
4. 14, 21. 5. 24, 30. 6. 20, 32. 7. 1.
8. 15, 10. 9. 6, 8. 10. 35, 42. 11. 4.
12. $\frac{ab}{a+b}$. 13. 50, 60, 90. 14. 0, 2 : 5.

XXXVI. 1. 14. 2. 18. 3. 15. 4. 12. 5. 4.
6. 4. 7. 2, $2\frac{1}{2}$. 8. 5. 9. 1, -1. 13. 45, 60, 80.
14. 4, 6, 9.

XXXVII. 1. 4. 3. 5 : 2. 4. 2. 5. 4.
6. 5. 7. 8. 8. abc. 9. $\frac{ac^2}{b^2}$. 10. £$113\frac{1}{3}$.
11. 15. 12. £15360.

XXXVIII. 1. 936. 2. $77\frac{1}{2}$. 3. 69. 4. $139\frac{1}{2}$.
5. $37\frac{1}{2}$. 6. -115. 7. 14, 16, 18. 8. $14\frac{1}{3}, 14\frac{2}{3}, \dots$
9. $6\frac{1}{2}, 5, \dots$ 10. $-\frac{1}{3}, \frac{1}{3}, \dots$ 11. 10, 4. 12. 82.
13. 5, 9, 13, 17. 14. 5, 7, 9. 15. 1, 2, 3, 4, 5.
16. 18, 19. 17. 7. 18. 5. 19. 1, 4, 7. 20. 1, 2.

XXXIX. 1. 1365. 2. $13\frac{4}{9}$. 3. $40\frac{3}{5}$. 4. $63(\sqrt{2}+1)$.
5. $\frac{665}{648}$. 6. $\frac{463}{96}$. 7. $\frac{3}{4}$. 8. $\frac{4}{3}$. 9. $\frac{2}{3}$. 10. $4\frac{1}{2}$.
11. $\frac{5}{33}$. 12. $\frac{41}{333}$. 13. $\frac{212}{495}$. 14. $\frac{557}{1980}$.
15. 4, 16, 64. 16. 8, 12, 18, 27. 17. -9; 27, -81, 243.
18. 3, 12, 48; or 36, -54, 81. 19. 1, 3, 9,... 20. 3, 6, 12.

XL. 1. $\frac{3}{2}, \frac{6}{5}, 1$. 2. $\frac{4}{5}, \frac{8}{13}, \frac{1}{2}$. 3. $3, \frac{12}{5}$.
4. $\frac{2}{15}, \frac{1}{12}, \frac{2}{33}$. 5. 6, 12. 6. 36, 64. 7. 1, 9. 8. 3, 9.

XLI. 1. 134596. 2. 5040. 3. 126. 4. 30240.
5. 11. 6. 1909. 7. 15504; 3876. 8. 27; 99.

XLII. 1. $a^{13}-13a^{12}x+78a^{11}x^2\dots-78a^2x^{11}+13ax^{12}-x^{13}$.
2. $243-810x^2+1080x^4-720x^6+240x^8-32x^{10}$.
3. $1-14y+84y^2-280y^3+560y^4-672y^5+448y^6-128y^7$.
4. $x^n+2nx^{n-1}y+2n(n-1)x^{n-2}y^2+\frac{4n(n-1)(n-2)}{3}x^{n-3}y^3$.
5. $1+4x+2x^2-8x^3-5x^4+8x^5+2x^6-4x^7+x^8$. 6. $1+5x+15x^2+30x^3+45x^4+51x^5+45x^6+30x^7+15x^8+5x^9+x^{10}$.
7. $1-8x+28x^2-56x^3+70x^4-56x^5+28x^6-8x^7+x^8$.
8. 5922. 9. 1590. 10. $x=2, y=3, n=5$. 11. $x=4, y=\frac{1}{2}, n=8$.
12. $a^{\frac{1}{4}}-\frac{a^{-\frac{3}{4}}x}{2}-\frac{3a^{-\frac{7}{4}}x^2}{8}-\frac{7a^{-\frac{11}{4}}x^3}{16}-\frac{77a^{-\frac{15}{4}}x^4}{128}$.
13. $1+\frac{x}{2}+\frac{x^2}{3}+\frac{13x^3}{54}$. 14. $1+2x+4x^2+8x^3+\dots$

15. $r+1$. 16. $\frac{3 \cdot 7 \cdot 11 \cdot 15 \cdot 19}{4^5 \lfloor 5} (3x)^{-\frac{3}{4}-5} y^5$.

17. $a^{-\frac{10}{3}} + 10a^{-\frac{13}{3}} b + 65a^{-\frac{16}{3}} b^2 + \frac{1040}{3} a^{-\frac{19}{3}} b^3 + \frac{4940}{3} a^{-\frac{22}{3}} b^4$.

18. $\frac{(r+1)(r+2)(r+3)}{1 \cdot 2 \cdot 3}$. 19. $1 + \frac{1}{2} x - \frac{3x^2}{8} - \frac{3x^3}{16}$.

20. $1 + \frac{x}{2} - \frac{x^2}{8} - \frac{7x^3}{16}$.

XLIII. 1. 2042132. 2. 22600. 3. 11101001010. 4. 2076. 5. $t4592$. 6. Radix 8. 7. Radix 6. 8. $9e21$; te. 9. Radix 5. 10. eee.

XLIV. 1. $\frac{bc}{a}$. 3. $n = \frac{1}{r}$.

MISCELLANEOUS. 1. 729, 369, 1, 41. 2. $41x - 51y$. 3. $9 - 30x + 37x^2 - 20x^3 + 4x^4$. 4. $1 + x - x^3 - x^4$, $1 - x + x^3 - x^4$. 5. $\frac{2x^2 + 3x - 4}{3x - 4}$. 6. $(4x^2 - 9)(9x^2 - 4)$.

7. $\frac{2}{a}$. 8. 3. 9. 240, 360. 10. £2, £$2\frac{2}{3}$.

11. $\frac{7x}{6} + \frac{7y}{6} + \frac{7z}{6}$, $x + \frac{13y}{6} + \frac{3z}{2}$. 12. 1. 13. $3b^2$.

14. $2x^2 - xy - 2y^2$. 15. $\frac{x^2 - x - 1}{x^2 + x + 1}$.

16. $(x-10)(x+1)(x+3)$. 17. $\frac{59}{(x-10)(x+1)(x+3)}$.

18. 5. 19. 7. 20. £40. 21. $2a - 2b - x - 2y$, $a + 3b + 4x + 4y$. 22. 11. 23. $x^4 - a^4$. 24. $\frac{x^2}{2} + \frac{x}{3} - \frac{1}{4}$.

25. $x^2 - 2$. 26. $\frac{ab - b^2}{b^2 - 4a^2}$. 27. $(16x^2 - 1)(x^2 - 4)$.

28. 6. 29. 14*s*, 21*s*, 52½*s*. 30. 100.

31. 1. 32. $(x^2 - a^2)(x^2 - b^2)$, $(x-a)(x-b)$.

33. $4x^4 - 2x^3y + x^2y^2 - xy^3 + \frac{y^4}{2}$. 34. $x - 2$.

35. $\frac{3(4x-y)}{2(3x^2+y^2)}$. 36. 1. 37. 4. 38. 2. 39. 30 minutes.

40. £18, £6. 41. $10x + 10z$. 42. $7x^2 - 2xy + y^2$, $-x^2 - 6xy + 7y^2$, $12x^4 - 10x^3y - x^2y^2 + 20xy^3 - 12y^4$.

43. $a + b - c$. 44. $x^2 + 1$. 45. $\frac{1}{x+1}$. 46. $(x^2-4)(x^2-9)$.

47. $\frac{x^2+x+2}{2x^2+x-1}$. 48. 1. 49. $\frac{16}{25}$. 50. 30 lbs.

51. $3a^4 - 5a^3b - 12a^2b^2 - ab^3 + 3b^4$, $3a^3 - 8a^2b - 4ab^2 + 3b^3$.

52. $2x - 5$. 53. 2. 54. $\frac{(a+b)a}{b}$. 55. 1; 2. 56. 3; 6.

57. 5; 8. 58. 4; 5; 2. 59. $\frac{a^2+b^2}{am+bn}$; $\frac{a^2+b^2}{bm-an}$.

60. $\frac{3}{5}$. 61. $x^4 + x^2 + 1 + \frac{1}{x^2} + \frac{1}{x^4}$.

62. $\frac{11x+2}{7x^2+7x+2}$. 63. $\frac{2ax}{x^2+1}$. 64. 1. 65. 4; 3.

66. 2; 4. 67. 3; -3. 68. 3. 69. 2.

70. 20; 40 years. 71. 1. 72. $\frac{2x-1}{2x-1}$, $\frac{8}{5}$.

73. $(x-2)(x-1)$, $(x-2)(x-5)$, $(x-1)(x-5)$. 74. 0.

75. $\frac{2}{5}$. 76. $\frac{17}{3}$, $\frac{4}{3}$. 77. 3 shillings, 2 shillings.

78. $3x^2 - x + 1$. 82. $(x^2 - 4y^2)^3$. 83. 3; -2.

84. 5. 85. 47 or 74. 86. 45 gallons.

87. $4x^3 - 3xy^2 + 5y^3$. 88. $52\frac{4}{5}$. 89. 4·85409. 91. $x - y$.

92. $8(x^2 + y^2)$; $48(x^4 - y^4)$. 93. $\frac{x^2+3y^2}{x^4-y^4}$. 94. 1. 95. 4, 5, 6.

96. 3, $-\frac{5}{3}$. 97. 20 miles. 98. Present price 3 pence per dozen. 99. $18\left(1 - \frac{1}{3^8}\right)$; 18. 100. 4, 8, 16.

101. x^4-1, $1+x^{\frac{1}{4}}+x^{\frac{1}{2}}$. 102. $(x^2-a^2)(x^6-a^6)$. 103. a.

104. 1. 105. 13, $\pm\sqrt{\frac{35}{11}}$. 106. ±3; ±4; ∓5; or ±3; ±5; ∓4. 107. 20 shillings.

108. 48. 109. $\frac{x}{y}+1-\frac{y}{x}$. 111. x^n+1+x^{-n}.

112. $\frac{x^2+5x+10}{x^3+2x^2+3x+6}$, $\frac{6}{7}$. 113. 7, $\frac{38}{9}$. 114. 1 or -3.

115. ±2; ±1. 116. $\frac{1}{3}$, $\frac{1}{2}$; $\frac{1}{2}$, $\frac{1}{3}$.

117. $\frac{2}{3}$. 118. $-x^{2}$. 119. 612.

121. $\frac{2a^2b^2-ab^3+a^2-3ab+b^2}{2ab^2-b^3+a-b}$. 122. $3x^2-5xy+2y^2$.

123. $x(3x+4)(x-6)$. 124. $\frac{2}{17}$. 125. 2, $\frac{1}{2}$.

126. 5, $-\frac{13}{4}$; 4, $\frac{5}{4}$. 127. 1, $\frac{5}{3}$; 2; $\frac{2}{3}$. 129. 3.

130. $3(3^n-1)$. 131. $\frac{1}{x}$. 132. $2x(3x+4)$.

133. 4, -3. 134. $x^2-x-6=0$. 135. $x^4=a^4$ or $\frac{1}{a^4}$.

136. ±2. 137. 8·19615. 138. $7-2\sqrt{3}$. 139. $\frac{x+y}{xy}$.

140. $\frac{c+b-2a}{b-a}$, $\frac{(a+c)(c+b-2a)}{2(b-a)}$. 141. 3, 2, 2.

142. 197, $3x^3-2x^2-5x-3$. 143. $a(a^2+b^2)$, $\frac{4xa}{x^2-a^2}$.

144. (1) 4. (2) 0, 5. (3) 5; 7. 145. (1) 3, $\frac{80}{11}$. (2) 8. (3) ±7; ±5. 146. 16; 16. 147. 20. 148. 16, 24.

149. $\frac{15}{4}$, 169. 150. As 5 to 1. 151. x^3. 152. a^2+4b.

153. $x-3$. 154. (1) 5. (2) 3. (3) 7; 4.

155. (1) 8. (2) 9. (3) ±9; ±7. 156. 30 pence.

157. 80. 158. £20. 159. $x+2a$.
161. a, $21a-27b+6c$, a^{11mnp}. 162. $3(a-x)$.
163. $72(x-y)^2(x^3+y^3)$. 164. (1) 9. (2) 8. (3) 12.
(4) 20; 2. 165. (1) 6, $\frac{2}{5}$. (2) 11. (3) ± 11, ± 13;
± 13, ± 11. (4) ± 2; ∓ 1. 166. 12 days. 167. 4, 8. 168. $\frac{4}{15}$.
170. 208; 400. 171. $23b-18a$. 172. 2, p^2, x^{pq}.
173. $x^3-3ax^2+3a^2x-a^3$. 174. (1) 13. (2) 4. (3) 6; 10.
(4) 3. 175. (1) 2, 4. (2) ± 5; ± 4. (3) ± 1, ± 7;
∓ 1, ± 5. (4) 1, 5; 5, 1. 176. $16\frac{4}{11}$ minutes after 12.
177. 36. 178. 40, 23. 179. $36\left(1-\frac{2^6}{3^6}\right)$, 36.
180. $7-\sqrt{6}$. 181. 15. 182. $\frac{3x+2}{x^2-2x-24}$. 184. (1) 9.
(2) 6; 8. (3) 4, $-\frac{7}{4}$. 185. (1) 13, -15. (2) 7. (3) 2, -1.
186. 288, 224. 187. 29 miles. 188. On the first day A won 8 games and lost 4 games. 190. $-85\frac{1}{2}$.
191. $\frac{18x^4+12x^3-43x^2+36x-18}{144}$, $\frac{6x^3-20x^2+x+36}{4}$.
192. $\frac{4x^2-15x+13}{x^3-6x^2+11x-6}$. 193. x^4-16y^4. 194. (1) 8.
(2) 7. (3) 40; 16. 195. (1) $\frac{5}{3}$, $-\frac{3}{2}$. (2) 13. (3) 2, 4;
4, 2. 196. 56 miles. 197. 24. 198. $23+15$.
199. a^2-ab+b^2, a^2+b^2. 200. 2, 4, 8, 16.
201. $\frac{1+9x-13x^2}{3(7-2x)}$. 202. x^2-2x+4. 203. $\frac{16x^2}{(2+3x)^2}$,
$\frac{2x}{1+x^2+x^4}$. 204. (1) 9. (2) $\frac{a^2}{b}$. (3) 6; 8.

205. (1) 7, $\frac{5}{6}$. (2) 1, -4. (3) ± 3; ± 2. 206. 10 miles.
207. 24. 208. 6 crowns + 18 shillings.
209. $2x^2+2ax+4b^2$. 210. 7, 11, 15, ...
211. $3x^3-2x^2y+3xy^2-5y^3$. 212. $-\dfrac{x^2}{12+5x-28x^2}$.
213. $\dfrac{4x^2-25x+37}{x^3-10x^2+31x-30}$. 214. (1) 9. (2) 16; 4.
(3) 3; 6; 9. 215. (1) 3, -6. (2) ± 7; ± 5. (3) 2, 4; 4, 2.
216. 114 of each. 217. 126. 218. 21. 219. 11, 12,
13, 14. 220. $3+2\sqrt{2}$. 221. x^6+x^3+1, px^2+qx-r.
222. $\dfrac{x^{m-1}}{b(a+bx)}$, $\dfrac{c+b+c}{a-b-c}$. 223. $(7x-4)(3x-2)(x^2-3)$.
224. (1) 9. (2) 23; 19. (3) 12; -24; 36. 225. (1) 23, -3.
(2) 100, -200. (3) $\dfrac{ac}{2a+2\sqrt{(a^2-b^2)}}$; $\dfrac{bc}{2a+2\sqrt{(a^2-b^2)}}$.
226. $\frac{7}{12}$ of a mile. 227. 500; 1000; 4000. 228. 2 hours;
4 hours. 229. 2, 5, 8, ... 230. $\frac{5n}{12}(9-n)$.
231. $x^{2a+2b+2c}$, $\dfrac{b(a^2+b^2)}{c(a^2-b^2)}$, $\dfrac{a^3+b^3}{(a-b)^2(a^2+b^2)}$.
232. $\dfrac{x+5}{9x^2-x-3}$. 233. (1) $\frac{1}{2}$. (2) $\pm\frac{1}{2}$. (3) $\frac{1}{4}$; $\frac{1}{5}$.
234. (1) 5, $\frac{27}{5}$. (2) $\dfrac{b-c\pm a}{\sqrt{a}}$. (3) 5; ± 4. 235. 19.
236. 150, 50. 237. 40, 50. 238. 1975.
239. $a^3+a^2b+ab^2+b^3$, $a+2b+3c$. 240. $x^2y^{\frac{1}{2}}+8x^{\frac{1}{2}}y^2$.
241. $14xy$, $\dfrac{2(1+x^2-x^3)}{1-x^4}$. 242. $x+a$. 243. 105 shillings.
244. 54. 245. 3, 5, 8. 246. $6\frac{1}{4}$ per cent.
247. 2200. 248. $\frac{5}{21}$. 249. 5·678, 1·234. 251. $2a-b$.

252. $a^{16}-x^{16}$, c. 253. $x(3a+2x)$. 254. (1) 5. (2) 114; 77. (3) 0, $\frac{a}{2}$. 255. 112; 96. 256. A has £5400, B has £7200. 257. 7; 13. 258. 80. 259. 8; 5. 260. £80. 261. c^2+2bc.

262. $x^{10}-1$, $-\frac{1}{x^2a^2}(2x^4+3ax^3-4a^2x^2-3a^3x+2a^4)$.

263. $x^2-x+1+\frac{1}{x}+\frac{1}{x^2}$. 264. $\frac{x^2-2x+3}{2x^2+5x-3}$, 1. 265. (1) $\frac{3}{7}$. (2) 1. (3) 18; 9. 266. (1) 3, -2. (2) 5, $\frac{6}{5}$. (3) 2, 3; 3, 2.

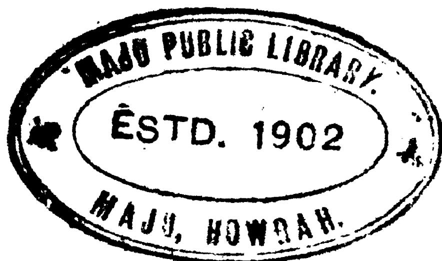
268. 45 shillings, 50 shillings. 270. $x^2+\frac{x}{2}-\frac{1}{3}$, $5-2\sqrt{2}$.

271. 0. 272. x^2+3x+8. 273. $\frac{3(a^2+x^2)}{a^2-x^2}$, $\frac{12a^2-8ax+5x^2}{15a^2+ax-2x^2}$. 274. $\frac{4x^4}{(4a-3x)(5a-2x)}$, $x(1-x)$.

275. $\frac{b(c+d)+1}{ab(c+d)+a+c+d}$, $\frac{a^2-ax+x^2}{a^2+ax+x^2}$. 276. (1) 2. (2) 11; 7. (3) 4; $\frac{16}{3}$. 277. (1) $\frac{a(a+b)}{a-b}$, $\frac{a(a-b)}{a+b}$. (2) 4, 7. (3) 5. 278. 7 + 7 miles. 281. $\frac{34}{35}$. 282. $2(a-b)(c-d)$, $-2bc$. 283. $\frac{x-6a}{x-11a}$, $-\frac{4xy}{x^2-y^2}$. 284. (1) 4. (2) 6; 4. (3) 5, $\frac{2}{3}$. 285. (1) $\frac{1}{2}$, $\frac{7}{5}$. (2) $2a$, $-a$, a, $-\frac{a}{2}$. (3) 1, $\frac{53}{19}$; -2, $\frac{47}{19}$. 286. Second boat 16 minutes. 287. 3 feet; 2 feet. 289. 18 feet. 290. $\frac{n}{2}\left\{\frac{2}{1+x}+\frac{(n-1)x}{1-x^2}\right\}$.

291. 0. 292. b^2. 294. $\frac{x^2+3}{x^2+2x+3}$.

295. (1) 4. (2) 61; 73. (3) 16; 8. 296. (1) 7, −8. (2) 7, $-\frac{29}{4}$. (3) 1, 5. 297. 144 minutes.
298. $4\frac{1}{2}$ hours with the stream, $7\frac{1}{2}$ hours against the stream. 299. $a-\frac{1}{2}b$, a, $a+\frac{1}{2}b$. 300. 3, −1.



THE END.

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